Local scale-invariance and ageing in noisy systems

Alan Picone and **Malte Henkel**

Laboratoire de Physique des Matériaux,[1] Université Henri Poincaré Nancy I,
B.P. 239, F – 54506 Vandœuvre lès Nancy Cedex, France

Abstract

The influence of the noise on the long-time ageing dynamics of a quenched ferromagnetic spin system with a non-conserved order parameter and described through a Langevin equation with a thermal noise term and a disordered initial state is studied. If the noiseless part of the system is Galilei-invariant and scale-invariant with dynamical exponent $z = 2$, the two-time linear response function is independent of the noise and therefore has exactly the form predicted from the local scale-invariance of the noiseless part. The two-time correlation function is exactly given in terms of certain noiseless three- and four-point response functions. An explicit scaling form of the two-time autocorrelation function follows. For disordered initial states, local scale-invariance is sufficient for the equality of the autocorrelation and autoresponse exponents in phase-ordering kinetics. The results for the scaling functions are confirmed through tests in the kinetic spherical model, the spin-wave approximation of the XY model, the critical voter model and the free random walk.

Keywords: conformal invariance, Schrödinger invariance, ageing, phase-ordering kinetics, Martin-Siggia-Rose theory, correlation function, response function

[1]Laboratoire associé au CNRS UMR 7556

1 Introduction

The study of ageing phenomena as they are known to occur in glassy and non-glassy systems presents one of the great challenges in current research into strongly coupled many-body systems far from thermal equilibrium. A common example of this kind of system is obtained as follows. Consider a magnet at a high-temperature initial state before quenching it to a final temperature T at or below its critical temperature $T_c > 0$. Then the temporal evolution of the system with T fixed is studied. A key insight has been the observation that many of the apparently erratic and history-dependent properties of such systems can be organized in terms of a simple scaling picture [90]. Underlying this phenomenological picture is the idea that the ageing phenomenon and the related slow evolution of the macroscopic observables comes from the slow motion of the domain walls which separate the competing correlated clusters. The domains are of a typical time-dependent size with length-scale $L(t)$, see [9, 11, 5, 39, 23, 19] for reviews. In recent years, much work has been performed on the ageing phenomena of simple ferromagnetic systems, in the hope that these systems might offer insight useful for the refined study also of ageing glassy materials. It has turned out that ageing is more fully revealed in two-time observables, such as the two-time (auto-)correlation function $C(t,s)$ or the two-time linear (auto-)response function $R(t,s)$ defined as

$$C(t,s) := \langle \phi(t)\phi(s)\rangle \ \ , \ \ R(t,s) := \left.\frac{\delta\langle\phi(t)\rangle}{\delta h(s)}\right|_{h=0} \tag{1.1}$$

where $\phi(t)$ denotes the time-dependent order parameter, $h(s)$ is the time-dependent conjugate magnetic field, t is referred to as *observation time* and s as *waiting time*. One says that the system undergoes *ageing* if C or R depend on both t and s and not merely on the difference $\tau = t - s$. According to the dynamical scaling alluded to above, one expects for times $t, s \gg t_{\rm micro}$ and $t - s \gg t_{\rm micro}$, where $t_{\rm micro}$ is some microscopic time scale, the following scaling forms

$$C(t,s) = s^{-b} f_C(t/s) \ \ , \ \ R(t,s) = s^{-1-a} f_R(t/s) \tag{1.2}$$

such that the scaling functions $f_{C,R}(y)$ satisfy the following asymptotic behaviour

$$f_C(y) \sim y^{-\lambda_C/z} \ \ , \ \ f_R(y) \sim y^{-\lambda_R/z} \tag{1.3}$$

as $y \longrightarrow \infty$ and where λ_C and λ_R, respectively, are known as the autocorrelation [27, 54] and autoresponse exponents [77], a and b are further non-equilibrium exponents and z is the dynamical exponent, where it has been tacitly assumed that the typical cluster size grows for late times as $L(t) \sim t^{1/z}$. The derivation of such growth laws from dynamical scaling has been studied in great detail [85]. For a non-conserved order parameter and $T < T_c$, the dynamical exponent $z = 2$. The exponents $\lambda_{C,R}$ are independent of the equilibrium exponents and of z [59, 39, 23]. Since a long time, the equality $\lambda_C = \lambda_R$ had been taken for granted but recently, examples to the contrary have been found for either long-ranged initial correlations in ageing ferromagnets [77] or else in the random-phase sine-Gordon (Cardy-Ostlund) model [89]. On the other hand, a second-order perturbative analysis of the time-dependent non-linear Ginzburg-Landau equation reproduces $\lambda_C = \lambda_R$ [67]. The precise relationship between λ_C and λ_R remains to be understood. If one uses an infinite-temperature initial state, one has $\lambda_C = \lambda_R \geq d/2$ [93].

For ageing ferromagnetic systems with a non-conserved order-parameter, the value of the exponent a depends on the properties of the equilibrium system as follows [47, 52]. A system is said to be in *class S* if its order-parameter correlator $C_{\rm eq}(\boldsymbol{r}) \sim \exp(-|\boldsymbol{r}|/\xi)$ with a finite ξ and it is said to be in *class L* if $C_{\rm eq}(\boldsymbol{r}) \sim |\boldsymbol{r}|^{-(d-2+\eta)}$, where η is a standard equilibrium critical exponent. Then

$$a = \begin{cases} 1/z & ; \text{ for class S} \\ (d-2+\eta)/z & ; \text{ for class L} \end{cases} \tag{1.4}$$

For example, in $d > 1$ dimensions, the kinetic Ising model with Glauber dynamics is in class S for temperatures $T < T_c$ and in class L at the critical temperature $T = T_c$. It is generally accepted that $b = 0$ for $T < T_c$ and $b = a$ if $T = T_c$, see e.g. [39].

The distance from equilibrium is conveniently measured through the *fluctuation-dissipation ratio* [20, 21]

$$X(t,s) := TR(t,s)\left(\frac{\partial C(t,s)}{\partial s}\right)^{-1} \tag{1.5}$$

At equilibrium, the fluctuation-dissipation theorem states that $X(t,s) = 1$. Ageing systems may also be characterized through the limit fluctuation-dissipation ratio

$$X_\infty = \lim_{s\to\infty}\left(\lim_{t\to\infty} X(t,s)\right) \tag{1.6}$$

Below criticality, one expects $X_\infty = 0$, but at $T = T_c$, the value of X_∞ should be universal according to the Godrèche-Luck conjecture [37, 38]. This universality has been confirmed in a large variety of systems in one and two space dimensions [38, 14, 51, 88]. The order of the limits is important, since $\lim_{t\to\infty}(\lim_{s\to\infty} X(t,s)) = 1$ always.

While these statements exhaust the content of dynamical scaling, it may be asked whether the form of the scaling functions $f_{C,R}(y)$ might be fixed in a generic, model-independent way through a generalization of that symmetry. Indeed, it has been shown [46] that an infinitesimal global scale-transformation $t \mapsto (1+\varepsilon)^z t$, $\boldsymbol{r} \mapsto (1+\varepsilon)\boldsymbol{r}$ with a constant ε such that $|\varepsilon| \ll 1$ can for any given value of z be extended to an infinitesimal *local scale-transformation* where now $\varepsilon = \varepsilon(t,\boldsymbol{r})$ may depend on both time and space.[2] It can be shown that the local scale-transformations so constructed act as dynamical symmetries of certain linear field equations which might be viewed as some effective renormalized equation of motion. Practically more important, assuming that the response functions of the theory transform covariantly under local scale-transformations, the exact form of the scaling function $f_R(y)$ is found [46, 45]

$$f_R(y) = r_0 y^{1+a-\lambda_R/z}\,(y-1)^{-1-a} \tag{1.7}$$

where r_0 is a normalization constant.[3] Indeed, in deriving this result one actually only requests that R transforms covariantly under under the *sub*algebra of the infinitesimal local scale-transformation which excluded time-translations. We say that a theory where the n-point functions built from certain 'quasiprimary' fields transform covariantly under an algebra of such extended scale-transformations is *locally scale-invariant* [46, 45]. The prediction (1.7) has been confirmed in a large class of ageing ferromagnets as reviewed in [46, 50]. The status of the scaling function $f_C(y)$ of the spin-spin correlator is less clear, however. Building on the Ohta-Jasnow-Kawasaki approximation (see [9]) Gaussian closure procedures [8, 83] in the O(n)-model produce approximate forms for $f_C(y)$ but we do not know of any other approach which does not involve some uncontrolled approximation.

Given the phenomenological success of (1.7), we wish to understand better where such a supposedly general and exact result derived from a dynamical symmetry and without using any model-specific properties might come from. In this paper, we shall concentrate on the important special case $z = 2$ which describes the phase-ordering kinetics after a quench to a temperature $T < T_c$ of a ferromagnet with a non-conserved order-parameter [85]. We recall that local scale-transformations are dynamical

[2] This extension of dynamical scaling has an analogue in critical equilibrium systems: there global scale invariance can be extended to conformal invariance, see [17, 24, 44] for introductions.

[3] In order to avoid misunderstandings, we recall that (1.7) holds for the total response function as defined in (1.1) *without* any subtractions meant to extract an 'ageing part'.

symmetries of certain differential equations, such as the free diffusion/free Schrödinger equation for $z = 2$. Indeed, the maximal dynamical symmetry of the free diffusion equation $\partial_t \phi = D\Delta\phi$ is known since a long time to be the so-called Schrödinger group [63, 41, 70], to be defined in section 2. Also, it is well-established that the same group also describes the dynamic symmetry of non-relativistic free-field theory [42, 57]. It also arises as dynamical symmetry in certain non-linear Schrödinger equations [31, 82, 62, 35, 81], the Burgers equation [60, 56] or the equations of fluid dynamics [74]. If in addition D is also considered as a variable, Schrödinger invariance in d dimension becomes a conformal invariance in $d+2$ dimensions [48]. The classification of non-linear equations and of systems of equations admitting as a dynamical symmetry the Schrödinger group or one of its subgroups (e.g. the Galilei group) has received a lot of mathematical attention, see [30, 32, 33, 34, 82, 18, 72]. The extension to dynamical exponents $z \neq 2$ needed for quenches to criticality or for glassy systems will be left for future work.

However, the setting just outlined is not yet sufficient for the description of ageing phenomena. Rather, we are interested in the time-dependent behaviour of spin systems coupled to a heat bath at temperature T. It is usually admitted that after coarse-graining, this may be modeled in terms of a Langevin equation. If there are no macroscopic conservation laws, the Langevin equation for the coarse-grained order parameter $\phi = \phi(t, \boldsymbol{r})$ should be model A in the Hohenberg-Halperin classification [53]

$$\frac{\partial \phi(t, \boldsymbol{r})}{\partial t} = -D\frac{\delta \mathcal{H}}{\delta \phi} + \eta(t, \boldsymbol{r}) \tag{1.8}$$

where $\mathcal{H}$ is the classical Hamiltonian, and D stands for the diffusion constant or equivalently some relaxation rate. Thermal noise is described by a Gaussian random force $\eta = \eta(t, \boldsymbol{r})$ and is thus characterized by its first two moments

$$\langle \eta(t, \boldsymbol{r})\rangle = 0 \ \ , \ \ \langle \eta(t, \boldsymbol{r})\eta(s, \boldsymbol{r}')\rangle = 2DT\,\delta(t-s)\,\delta(\boldsymbol{r}-\boldsymbol{r}') \tag{1.9}$$

where T is the bath temperature. It is well-known [53, 9] that this formalism describes the relaxation of the system towards its equilibrium state given by the probability distribution $P_{\rm eq} \sim e^{-\mathcal{H}/T}$. In addition, initial conditions must be taken into account and are described in terms of the initial correlation function

$$a(\boldsymbol{r}-\boldsymbol{r}') := C(0, 0; \boldsymbol{r}, \boldsymbol{r}') := \langle \phi(0, \boldsymbol{r})\phi(0, \boldsymbol{r}')\rangle \tag{1.10}$$

and where we already anticipated spatial translation invariance.

Neither the thermal noise nor the initial correlations described by $a(\boldsymbol{r})$ are included into the local scale-transformations as studied in [46] which come from systems such as the free diffusion equation. In this paper, we want to show how both these sources of fluctuations may be taken into account and we shall explicitly derive the two-time response and correlation functions. Our analysis will be restricted to the case where $z = 2$ which for instance is already enough to describe ageing below criticality.

As we shall show, it is useful to slightly generalize the problem and to consider the kinetics of systems which in the simplest case may be described by a quadratic Hamiltonian of the form

$$\mathcal{H}[\phi] = \frac{1}{2}\int \mathrm{d}t\mathrm{d}\boldsymbol{r}\ \left[\left(\frac{\partial \phi}{\partial \boldsymbol{r}}\right)^2 + v(t)\phi^2\right] \tag{1.11}$$

where $v(t)$ is a time-dependent external potential. Formally, at the level of relativistic free-field theory, $v(t)$ corresponds to a (time-dependent) mass squared which would measure the distance from a critical point. Alternatively $v(t)$ may be viewed as a Lagrange multiplier in order to ensure the constraint $\langle \phi(t, \boldsymbol{r})\phi(t, \boldsymbol{r})\rangle = 1$ and we shall make this explicit through the example for the kinetic spherical model in section 5. In a physically more appealing way, time-dependent potentials arise when a many-body

system is brought into contact with a heat bath whose temperature $T(t)$ is time-dependent [78]. In this paper, we shall be interested in the dynamics symmetries of Langevin equations derived from a free-field Hamiltonian (1.11). In particular, we shall compare the situation without (i.e. $T = 0$) and with thermal noise (i.e. $T > 0$). For simplicity, we shall refer to all equations (1.8) obtained from the Hamiltonian (1.11) as 'free Schrödinger equations'.

The study of the dynamic symmetries of such free-field theories will yield useful insights which we expect to extend to physically more realistic interacting field theories where $\mathcal{H}$ would also contain higher than merely quadratic terms. If we identify $D^{-1} = 2im$, it is clear that the order parameter ϕ is given by a noisy Schrödinger equation in an external time-dependent potential $v(t)$.

This paper is organized as follows. In section 2, we first review the basics of Schrödinger-invariance in the absence of thermal noise and without initial correlations and then show that through a gauge-transformation involving $v(t)$, the entire phenomenology of ageing and in particular (1.7) can be reproduced. We also consider the selection rules which follow from Galilei-invariance. In section 3, after having reformulated the problem in terms of the field-theoretic Martin-Siggia-Rose formalism, we study the effects of thermal noise and/or initial correlations on free-field theory given by a Hamiltonian (1.11). In section 4, these results are extended to any field theory with for $T = 0$ and $a(\boldsymbol{r}) = 0$ is Galilei-invariant. We find that the two-time response function R is independent of both T and $a(\boldsymbol{r})$ and obtain a new reduction formula (4.9) which relates C to certain three- and four-point response functions to be evaluated in the noiseless theory and discuss the scaling of the resulting two-time auto-correlation function. In sections 5-7, these results are tested in several exactly solvable systems (with an underlying free-field theory) undergoing ageing with $z = 2$, namely the kinetic spherical model, the XY-model in spin-wave approximation, the critical voter model and the free random walk. We conclude in section 8. Appendix A deals with technical aspects of Gaussian integration and appendix B analyses a special four-point response function. In appendix C we consider a generalized realization of local scale-invariance and its application to the $1D$ Glauber-Ising model

2 Local scale-invariance: a reminder

2.1 Schrödinger-invariance

We begin by reviewing the kinematic symmetries of Schrödinger equations with a time-dependent potential, but without a noise term. A long time ago, Niederer [71] obtained the maximal kinematic symmetry group of the Schrödinger equation for an arbitrary potential $v = v(t, \boldsymbol{r})$ and he also gave a few examples where that group is isomorphic to the maximal kinematic group $Sch(d)$ of the free Schrödinger equation [63, 70]. The group $Sch(d)$ is called the *Schrödinger group* [70].

We recall the definition of $Sch(d)$. On the time and space coordinates $(t, \boldsymbol{r})$ it acts as $(t, \boldsymbol{r}) \mapsto (t', \boldsymbol{r}') = g(t, \boldsymbol{r})$ where

$$t \longmapsto t' = \frac{\alpha t + \beta}{\gamma t + \delta} \ \ , \ \ \boldsymbol{r} \longmapsto \boldsymbol{r}' = \frac{\mathcal{R}\boldsymbol{r} + \boldsymbol{v}t + \boldsymbol{a}}{\gamma t + \delta} \ \ ; \ \ \alpha\delta - \beta\gamma = 1 \tag{2.1}$$

where $\mathcal{R}$ is a rotation matrix. The action of $Sch(d)$ on the space of solutions ϕ of the free Schrödinger equation is projective, that is, the wave function $\phi = \phi(t, \boldsymbol{r})$ transforms into

$$\phi(t, \boldsymbol{r}) \longmapsto (T_g\phi)(t, \boldsymbol{r}) = f_g[g^{-1}(t, \boldsymbol{r})]\phi[g^{-1}(t, \boldsymbol{r})] \tag{2.2}$$

and the companion function f_g is explicitly known [70, 71]. The projective unitary irreducible representations of $Sch(d)$ are classified [76]. We now carry this over to field theory and consider fields

transforming according to (2.2). By analogy with an analogous terminology in conformal field theory [3], a field ϕ transforming according to (2.2) and with $g(t,\boldsymbol{r})$ given by (2.1) is called *quasiprimary* [46]. Schrödinger-invariance is the $z=2$ special case of local scale-invariance, with time-translations added.

Besides the examples given in [71], there exist further noiseless Schrödinger equations with a maximal kinematic group isomorphic to *Sch*(d). Consider the noiseless Langevin equation

$$D^{-1}\frac{\partial\phi(t,\boldsymbol{r})}{\partial t} = \Delta\phi(t,\boldsymbol{r}) - v(t)\phi(t,\boldsymbol{r}) \tag{2.3}$$

where D is the diffusion constant. This equation can be reduced to the free Schrödinger equation $D^{-1}\partial_t\Psi(t,\boldsymbol{r}) = \Delta\Psi(t,\boldsymbol{r})$ through the gauge transformation

$$\phi(t,\boldsymbol{r}) = \Psi(t,\boldsymbol{r})\exp\left(-D\int_0^t \mathrm{d}u\, v(u)\right) \tag{2.4}$$

Since the kinematic symmetries of the free Schrödinger equation are well-understood and the realization of the Schrödinger group is explicitly known, the corresponding realization for the case at hand, similar to (2.2), readily follows.

It turns out that the only change occurs in the companion function f_g. Let $f_g^{(0)}$ stand for the companion function of the free Schrödinger equation, then because of the gauge transformation eq. (2.4) we find

$$f_g(t,\boldsymbol{r}) = f_g^{(0)}(t',\boldsymbol{r})\exp\left(-D\int_t^{t'} \mathrm{d}u\, v(u)\right) \tag{2.5}$$

where $t'=t'(t)$ has been defined in eq. (2.1). The generators of the Lie algebra $\mathfrak{sch}_1$ of this realization of *Sch*(1), appropriate for the equation (2.3) with the potential $v(t)$, read

$$\begin{aligned}
X_{-1} &= -\partial_t + \frac{v(t)}{2\mathcal{M}} && \text{time drift}\\
X_0 &= -t\partial_t - \frac{1}{2}r\partial_r - \frac{x}{2} + t\frac{v(t)}{2\mathcal{M}} && \text{dilatation}\\
X_1 &= -t^2\partial_t - tr\partial_r - xt + t^2\frac{v(t)}{2\mathcal{M}} - \frac{\mathcal{M}}{2}r^2 && \text{special Schrödinger transformation}\\
Y_{-1/2} &= -\partial_r && \text{space translation}\\
Y_{1/2} &= -t\partial_r - \mathcal{M}r && \text{Galilei transformation}\\
M_0 &= -\mathcal{M} && \text{phase shift.}
\end{aligned} \tag{2.6}$$

where we expressed the diffusion constant $D^{-1}=2\mathcal{M}$ as the 'mass' $\mathcal{M}$ and x denotes the scaling dimension of the wave function $\phi(t,\boldsymbol{r})$. Of course, $x=d/2$ for a solution of the free Schrödinger equation, but it will be useful to consider arbitrary values of x as well.

The non-vanishing commutators of the Lie algebra $\mathfrak{sch}_1$ spanned by the generators (2.6) are

$$[X_n, X_{n'}] = (n-n')X_{n+n'} \;\;,\;\; [X_n, Y_m] = \left(\frac{n}{2}-m\right)Y_{n+m} \;\;,\;\; \left[Y_{1/2}, Y_{-1/2}\right] = M_0 \tag{2.7}$$

where $n,n'\in\{\pm 1, 0\}$, $m\in\{\pm\frac{1}{2}\}$ and with straightforward extensions to $d>1$, see [46].

2.2 Galilei-covariance of correlators

When discussing the dynamic symmetries of a time-dependent statistical system, the requirement of Galilei-invariance plays a particular rôle. Indeed, for a system with local interactions and which is

invariant under space translations, scale transformations with a dynamical exponent $z = 2$ and in addition Galilei-invariant, it can be shown that there exists a Ward identity such that the system is also invariant under the 'special' Schrödinger transformation (2.6) [42, 43, 48].

We shall be particularly interested in the two-point correlator C and the linear response function R built from the order parameter $\phi(t, \boldsymbol{r})$. Using Martin-Siggia-Rose theory (MSR theory) which we shall briefly review in section 3, these may be expressed in terms of ϕ and the so-called response field $\widetilde{\phi}$ as follows

$$\begin{aligned} C(t, s; \boldsymbol{r}, \boldsymbol{r}') &:= \langle \phi(t, \boldsymbol{r})\phi(s, \boldsymbol{r}')\rangle \\ R\,(t, s; \boldsymbol{r}, \boldsymbol{r}') &:= \left.\frac{\delta\langle\phi(t, \boldsymbol{r})\rangle}{\delta h(s, \boldsymbol{r}')}\right|_{h=0} = \langle \phi(t, \boldsymbol{r})\widetilde{\phi}(s, \boldsymbol{r}')\rangle \\ \widetilde{C}\,(t, s; \boldsymbol{r}, \boldsymbol{r}') &:= \langle \widetilde{\phi}(t, \boldsymbol{r})\widetilde{\phi}(s, \boldsymbol{r}')\rangle \end{aligned} \tag{2.8}$$

where h is the magnetic field conjugate to the order parameter ϕ. Later, we shall often refer to t as the *observation time* and to s as the *waiting time*.

Generalizing the above definition to $v(t) \neq 0$, we say that a field ϕ is *quasiprimary* if its infinitesimal change under $\mathfrak{sch}_1$ is given by the generators (2.6) with $v(t) \neq 0$. A quasiprimary field ϕ is characterized by its scaling dimension x and its 'mass' $\mathcal{M} \geq 0$. In turn, if the response field $\widetilde{\phi}$ associated to ϕ is also quasiprimary, it has a scaling dimension denoted by $\widetilde{x}$ and the 'mass'

$$\widetilde{\mathcal{M}} = -\mathcal{M} \leq 0 \tag{2.9}$$

This important fact will be used later on. The argument leading to the result (2.9) was discussed in detail in [48] and will not be repeated here.

If both ϕ and $\widetilde{\phi}$ transform as quasiprimary fields of a Schrödinger-invariant theory, the generators eqs. (2.6) can be used to derive restrictions on the form of any multipoint correlator and in particular determine the two-point functions completely. If $\mathcal{X}_i$ is any of the generators of $\mathfrak{sch}_d$ acting on the i^{th} particle in a n-point correlator $\mathfrak{A}\{t_i, \boldsymbol{r}_i\}$ where $i = 1, \ldots, n$ (see (2.8) for $n = 2$), we have a set of differential equations

$$(\mathcal{X}_1 + \ldots + \mathcal{X}_n)\,\mathfrak{A}\{t_i, \boldsymbol{r}_i\} = 0 \tag{2.10}$$

If rotation invariance can be assumed (and we shall implicitly do so throughout this paper), for the calculation of the two-point functions it is enough to consider the one-dimensional case and use the generators of eq. (2.6). Then a straightforward calculation [80] gives, provided the 'mass' of the order parameter is positive $\mathcal{M} > 0$, see e.g. [43, 46]

$$\begin{aligned} \widetilde{C}_0\,(t, s; \boldsymbol{r}, \boldsymbol{r}') = 0 \\ C_0\,(t, s; \boldsymbol{r}, \boldsymbol{r}') = 0 \end{aligned} \tag{2.11}$$

whereas the response function is basically the gauge-transformed expression of the well-known zero-potential Gaussian response $\mathcal{R}$, i.e.

$$\begin{aligned} R_0\,(t, s; \boldsymbol{r}, \boldsymbol{r}') &= \frac{k(t)}{k(s)}\mathcal{R}\,(t, s; \boldsymbol{r}, \boldsymbol{r}') \\ \mathcal{R}\,(t, s; \boldsymbol{r}, \boldsymbol{r}') &= \delta_{x,\widetilde{x}}\, r_0\, \Theta(t-s)\,(t-s)^{-x} \exp\left(-\frac{\mathcal{M}}{2}\frac{(\boldsymbol{r}-\boldsymbol{r}')^2}{(t-s)}\right) \\ k(t) &:= \exp\left(-\frac{1}{2\mathcal{M}}\int^t \mathrm{d}u\, v(u)\right) \end{aligned} \tag{2.12}$$

where r_0 is a normalization constant and Θ is the Heaviside function which expresses causality. As they stand, eqs. (2.11,2.12) hold for $T = 0$ and we shall from now on use the index 0 to remind the reader of this fact.

On the other hand, if the system is not rotation-invariant, we can repeat the same argument in any fixed direction of space and the non-universal constant $\mathcal{M}$ becomes direction-dependent. Indeed, rotation invariance is broken for phase-ordering systems defined on a lattice [86, 87] for sufficiently small temperatures. Even then, local scale invariance still holds in every single space direction, as exemplified in the $2D$ and $3D$ Ising models with Glauber dynamics [49].

A few comments are in order.

1. Eqs. (2.11,2.12) provide a manifest example of the superselection rule of Galilei invariance, also known as Bargman superselection rule [2]. Explicitly, if $\Phi_i(t_i, \boldsymbol{r}_i)$ are Galilei-covariant fields, each with a 'mass' $\mathcal{M}_i$, Galilei-covariance implies [2, 43]

$$\langle \Phi_1(t_1, \boldsymbol{r}_1) \ldots \Phi_n(t_n, \boldsymbol{r}_n) \rangle = \delta_{\mathcal{M}_1 + \ldots + \mathcal{M}_n, 0}\, F\left(\{t_i, \boldsymbol{r}_i\}\right) \tag{2.13}$$

   By physical convention, the 'masses' of the fields ϕ are non-negative, viz. $\mathcal{M}_i \geq 0$. Furthermore, the response fields $\widetilde{\phi}$ have *negative* 'masses' $\widetilde{\mathcal{M}}_i \leq 0$ and the result (2.11) follows.

2. In the introduction, we reviewed the result (1.7) for the autoresponse function, derived for arbitrary z from a generalization of Schrödinger invariance [46, 45]. For $z = 2$, eq. (1.7) coïncides with our result (2.12) provided that

$$\begin{aligned} x &= \widetilde{x} = 1 + a \\ v(t) &= (2\mathcal{M})\, \frac{1 + a - \lambda_R/2}{t} \end{aligned} \tag{2.14}$$

3. However, there is an important difference in our derivation of the scaling form of $R_0(t, s)$ with respect to [46, 45]: because time-translation invariance is broken in ageing phenomena, covariance of R_0 was required to hold merely under a subalgebra of $\mathfrak{sch}_d$ where the time-translations were excluded. Indeed, in [48], such subalgebras were studied systematically and a relationship with the parabolic subalgebras of the (complexified) conformal algebra $\mathfrak{conf}_{d+2}$ was found. On the other hand, the realization of $\mathfrak{sch}_d$ used in [46, 45, 48] applies to a vanishing potential $v(t) = 0$.

   Here, we do not follow that point of view. We consider the more general realization of the entire algebra $\mathfrak{sch}_1$ with a time-dependent potential $v(t)$ and require that both ϕ and $\widetilde{\phi}$ transform as quasiprimary fields under the *whole* set of generators (2.6).

   That these two different approaches, given the conditions (2.14), yield the *same* phenomenology of the scaling of the two-time autoresponse function is our first result and will be crucial for the developments to follow.

4. At first sight, the result (2.11) that the two-time autocorrelator $C(t, s) = \langle \phi(t, \boldsymbol{r}) \phi(s, \boldsymbol{r}) \rangle = 0$ may appear strange and indeed does not hold true in concrete models. In the next sections, we shall show that this apparent contradiction comes from the fact that the noiseless Schrödinger equation does not take the thermal noise into account. As we shall see, the reformulation of Schrödinger invariance in ageing systems in terms of a noisy Schrödinger equation with a time-dependent potential allows to arrive at physically meaningful predictions for correlation functions. Explicit confirmations in exactly soluble models will be presented.

3 Response and correlation functions for non-interacting Gaussian theories

3.1 The Martin-Siggia-Rose formalism

It is useful to treat noisy Langevin equations in the context of the Martin-Siggia-Rose (MSR) formalism [66, 59, 94, 23, 91]. In equilibrium, the integration over the Gaussian noise η can be carried out by introducing a response field $\widetilde{\phi}$.[4] It can be shown that the stochastic Langevin equation (1.8) can be obtained from the following effective action $\Sigma[\phi,\widetilde{\phi}]$

$$\Sigma[\phi,\widetilde{\phi}] = \int \mathrm{d}t\,\mathrm{d}\boldsymbol{r}\;\widetilde{\phi}\left(\frac{\partial\phi}{\partial t}+D\frac{\delta\mathcal{H}}{\delta\phi}\right) - \frac{1}{2}\int \mathrm{d}t\,\mathrm{d}\boldsymbol{r}\,\mathrm{d}t'\,\mathrm{d}\boldsymbol{r}'\;\langle\eta(t,\boldsymbol{r})\eta(t',\boldsymbol{r}')\rangle\;\widetilde{\phi}(t,\boldsymbol{r})\widetilde{\phi}(t',\boldsymbol{r}') \tag{3.1}$$

This action appears in the generating functional $Z=\int\mathcal{D}\phi\mathcal{D}\widetilde{\phi}\;e^{-\Sigma[\phi,\widetilde{\phi}]}$ expressed as a path integral. In this way, the original dynamical problem in d dimensions has been mapped onto one of statistical mechanics in $d+1$ dimensions.

As long as one is merely interested in equilibrium behaviour and provided the dynamics is ergodic, there is no need to worry about initial conditions, which might be said to be specified at a time $t=-\infty$. Here, we are interested in how the equilibrium state is reached from a given initial state and must include into the action a term describing the initial preparation of the system. One has

$$S[\phi,\widetilde{\phi}] = \Sigma[\phi,\widetilde{\phi}] + \sigma[\phi,\widetilde{\phi}] \tag{3.2}$$

where $\Sigma[\phi,\widetilde{\phi}]$ (3.1) describes the 'bulk' evolution of the system as derived from the Langevin equation while $\sigma[\phi,\widetilde{\phi}]$ describes the initial conditions at time $t=0$. As already pointed out by Mazenko [67], it may be written as

$$\sigma[\phi,\widetilde{\phi}] = -\frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d}\mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{r}'\;\widetilde{\phi}(0,\boldsymbol{r})a(\boldsymbol{r}-\boldsymbol{r}')\widetilde{\phi}(0,\boldsymbol{r}') \tag{3.3}$$

where it is implicitly admitted that $\langle\phi(0,\boldsymbol{r})\rangle=0$ and $a(\boldsymbol{r})$ is the initial two-point correlator

$$a(\boldsymbol{r}) := C(0,0;\boldsymbol{r}+\boldsymbol{r}',\boldsymbol{r}') = \langle\phi(0,\boldsymbol{r}+\boldsymbol{r}')\phi(0,\boldsymbol{r}')\rangle \tag{3.4}$$

From spatial translation-invariance, it follows that $a(\boldsymbol{r})=a(-\boldsymbol{r})$ which we shall admit throughout.

We call the theory described by the action S_0 alone the *noiseless* theory.

For a free field, the noiseless and the thermal parts of the MSR action read (we also have set $D=1$)

$$\begin{aligned}
S_0[\phi,\widetilde{\phi}] &:= \int\mathrm{d}t\,\mathrm{d}\boldsymbol{r}\;\widetilde{\phi}\left(\frac{\partial\phi}{\partial t}-\Delta\phi+v(t)\phi\right)\\
\mathcal{S}[\phi,\widetilde{\phi}] &:= -T\int\mathrm{d}t\,\mathrm{d}\boldsymbol{r}\;\widetilde{\phi}^2(t,\boldsymbol{r})\\
\Sigma[\phi,\widetilde{\phi}] &= S_0[\phi,\widetilde{\phi}]+\mathcal{S}[\phi,\widetilde{\phi}]
\end{aligned} \tag{3.5}$$

We shall refer to the contribution described by $\mathcal{S}$ and σ as the *thermal* and *initial* noise, respectively.

[4] In the systematic terminology of [17], this should be rather called a *response operator*, because $\widetilde{\phi}$ will become an operator in a canonical quantization scheme of the action. The notion of a response field should have been reserved to the canonically conjugate variable of $\widetilde{\phi}$. Since we shall not use the operator formalism here, we shall simply, but sloppily, talk about $\widetilde{\phi}$ as a response field.

We point out that field-theoretic studies of critical dynamics use a different initial term, namely [58]

$$\sigma_c[\phi,\widetilde{\phi}] = \frac{\tau_0}{2}\int_{\mathbb{R}^d} \mathrm{d}\boldsymbol{r}\ \left(\phi(0,\boldsymbol{r}) - m(\boldsymbol{r})\right)^2 \tag{3.6}$$

This specifies an initial macroscopic state with spatially varying order parameter $\langle\phi(0,\boldsymbol{r})\rangle = m(\boldsymbol{r})$ and spatial correlations decaying on a finite scale proportional to ${\tau_0}^{-1}$. Ageing *at* criticality was studied in the O(n)-model using the ε-expansion with the initial term σ_c [58, 12, 13, 14, 15]. However, the use of σ_c instead of σ for temperatures below criticality would lead to contradictions.

Treating the noisy Langevin equation (1.8) as the classical equation of motion of the field-theory (MSR) action eqs. (3.2,3.5,3.3) has the following advantages.

- Thermal fluctuations and initial conditions are explicitly included. To emphasize this, notice that the noisy contributions to the MSR action are

$$\begin{aligned} S[\phi,\widetilde{\phi}] - S_0[\phi,\widetilde{\phi}] &= -\int \mathrm{d}u\,\mathrm{d}\boldsymbol{r}\,\mathrm{d}u'\,\mathrm{d}\boldsymbol{r}'\ \widetilde{\phi}(u,\boldsymbol{r})\kappa(u,u';\boldsymbol{r}-\boldsymbol{r}')\widetilde{\phi}(u',\boldsymbol{r}') \\ \kappa(u,u';\boldsymbol{r}) &= T\delta(u-u')\delta(\boldsymbol{r}) + \frac{1}{2}\delta(u)\delta(u')a(\boldsymbol{r}) \end{aligned} \tag{3.7}$$

  where κ includes both the effects of thermal and of initial-state fluctuations.

- The response field $\widetilde{\phi}$ describes the thermal noise as can be seen from the equations of motion derived from the free-field MSR action (3.5)

$$\begin{aligned} \frac{\partial\phi(t,\boldsymbol{r})}{\partial t} &= \Delta\phi(t,\boldsymbol{r}) - v(t)\phi(t,\boldsymbol{r}) + 2T\widetilde{\phi}(t,\boldsymbol{r}) \\ -\frac{\partial\widetilde{\phi}(t,\boldsymbol{r})}{\partial t} &= \Delta\widetilde{\phi}(t,\boldsymbol{r}) - v(t)\widetilde{\phi}(t,\boldsymbol{r}) \end{aligned} \tag{3.8}$$

  The first equation (3.8) reduces to the Langevin equation (1.8) provided one makes the formal identification

$$\eta(t,\boldsymbol{r}) = 2T\widetilde{\phi}(t,\boldsymbol{r}) \tag{3.9}$$

  Therefore, at the classical level, the stochastic Langevin equation eq. (1.8) is described by two deterministic equations. In our case they are both of Schrödinger type and with opposite masses for the field ϕ and the response field $\widetilde{\phi}$.

- Averages of any n-point function built from the fields $\phi,\widetilde{\phi}$ can be expressed in terms of the functional integral

$$\left\langle F\{\phi(t_i,\boldsymbol{r}_i),\widetilde{\phi}(t_j,\boldsymbol{r}_j)\}\right\rangle = \int\mathcal{D}\phi\,\mathcal{D}\widetilde{\phi}\ \exp\left(-S[\phi,\widetilde{\phi}]\right) F\{\phi(t_i,\boldsymbol{r}_i),\widetilde{\phi}(t_j,\boldsymbol{r}_j)\} \tag{3.10}$$

  with the normalization $\langle 1\rangle = 1$.

  For example adding a magnetic perturbation $\delta\mathcal{H}_{\mathrm{mag}} = -h\phi$ to the Hamiltonian (1.11) and then computing the mean of the order-parameter to first order in h, the relation $R(t,s;\boldsymbol{r},\boldsymbol{r}') = \langle\phi(t;\boldsymbol{r})\widetilde{\phi}(s,\boldsymbol{r}')\rangle$ of eq. (2.8) is easily reproduced.

- The use of Martin-Siggia-Rose formalism makes the machinery of the field-theoretic renormalization-group available [94, 17, 91] but we shall not pursue this here.

It will be useful to split the calculation of averages into two steps as follows

$$
\begin{aligned}
\left\langle F\{\phi(t_i,\boldsymbol{r}_i),\widetilde{\phi}(t_j,\boldsymbol{r}_j)\}\right\rangle &= \left\langle F\{\phi(t_i,\boldsymbol{r}_i),\widetilde{\phi}(t_j,\boldsymbol{r}_j)\}\exp\left(-\mathcal{S}[\phi,\widetilde{\phi}]-\sigma[\phi,\widetilde{\phi}]\right)\right\rangle_0 \\
\left\langle F\{\phi(t_i,\boldsymbol{r}_i),\widetilde{\phi}(t_j,\boldsymbol{r}_j)\}\right\rangle_0 &= \int \mathcal{D}\phi\,\mathcal{D}\widetilde{\phi}\,\exp\left(-S_0[\phi,\widetilde{\phi}]\right)F\{\phi(t_i,\boldsymbol{r}_i),\widetilde{\phi}(t_j,\boldsymbol{r}_j)\}
\end{aligned} \tag{3.11}
$$

where the notation $\langle\rangle_0$ (and more generally the index 0) refers from now on to averages of the non-fluctuating theory. This allows to make use of the Schrödinger invariance of the noiseless theory.

3.2 Analytical results for free fields

In this section, we find both response and correlation functions for free-field Martin-Siggia-Rose theory, as given by eqs. (3.2,3.3,3.5).

3.2.1 Two-point functions without noise

The free Martin-Siggia-Rose action $S[\phi,\widetilde{\phi}]$ has a Gaussian structure. We shall write it as

$$
S[\phi,\widetilde{\phi}] = \int \mathrm{d}u\,\mathrm{d}\boldsymbol{r}\,\mathrm{d}u'\,\mathrm{d}\boldsymbol{r}'\;\Phi(u;\boldsymbol{r})^T\mathcal{Q}(u,u';\boldsymbol{r},\boldsymbol{r}')\Phi(u';\boldsymbol{r}') \tag{3.12}
$$

where $\Phi = \left(\begin{array}{c}\phi\\ \widetilde{\phi}\end{array}\right)$ is the two-component field built from ϕ and $\widetilde{\phi}$, and Φ^T stands for its transpose. The kernel $\mathcal{Q}$ reads

$$
\mathcal{Q}(u,u';\boldsymbol{r},\boldsymbol{r}') = \frac{1}{2}\left[\begin{array}{cc} 0 & \delta(u-u')\delta(\boldsymbol{r}-\boldsymbol{r}')\left(-\Delta-\partial_u\right)+\frac{1}{2}v(u) \\ \delta(u-u')\delta(\boldsymbol{r}-\boldsymbol{r}')\left(-\Delta+\partial_u\right)+\frac{1}{2}v(u) & -2\kappa(u,u';\boldsymbol{r}-\boldsymbol{r}') \end{array}\right] \tag{3.13}
$$

This peculiar form of the Lagrangian density is quite suggestive as regards the Galilei-invariance. When $\kappa=0$, that is in absence of noise, the quadratic form $\mathcal{Q}$ is antidiagonal. This is one way of presenting the Bargman superselection rules and leads in particular to $\langle\phi\phi\rangle = \langle\widetilde{\phi}\widetilde{\phi}\rangle = 0$ which is a manifestation of Galilei-invariance of the noiseless system. The presence of noise just breaks this symmetry.

In order to study systematically the rôle of the noise, we shall expand around the non-fluctuating theory. The correlation functions $C_0, \widetilde{C}_0$ and the linear response function R_0 are

$$
\begin{aligned}
C_0(t,s;\boldsymbol{r},\boldsymbol{r}') &= 0 \\
\widetilde{C}_0(t,s;\boldsymbol{r},\boldsymbol{r}') &= 0 \\
R_0\left(t,s;\boldsymbol{r},\boldsymbol{r}'\right) &= \frac{k(t)}{k(s)}\Theta(t-s)\left(4\pi(t-s)\right)^{-d/2}\exp\left(-\frac{1}{4}\frac{(\boldsymbol{r}-\boldsymbol{r}')^2}{(t-s)}\right)
\end{aligned} \tag{3.14}
$$

This follows since the quadratic form $\mathcal{Q}$ is antidiagonal the field ϕ can only be coupled to $\widetilde{\phi}$ and the result is just the bare propagator of the theory. It is clear that only the antidiagonality of $\mathcal{Q}$ is important to derive this result and the explicit free-field form (3.12) is not required for (3.14) to hold.

These results fully agree with the Schrödinger-invariance prediction eq. (2.12,2.11) with the identifications $x=\widetilde{x}=d/2$, $\mathcal{M}=1/2$ and $r_0=(4\pi)^{-d/2}$.

A further manifestation of the Galilei invariance of the noiseless theory is the fact that

$$\langle \underbrace{\phi\cdots\phi}_{n}\ \underbrace{\widetilde{\phi}\cdots\widetilde{\phi}}_{m}\rangle_0 = 0 \tag{3.15}$$

unless $n = m$ as is easily checked. This is a further example of the Bargman superselection rule (2.13) and will be important in what follows.

3.2.2 Two-point functions in presence of noise

We now find the same two-point functions in the presence of noise.

We begin with the response functions $R(t,s;\boldsymbol{r};\boldsymbol{r'})$ which is found from (3.14) by averaging with the noiseless weight $\exp\left(-S_0[\phi,\widetilde{\phi}]\right)$

$$R(t,s;\boldsymbol{r},\boldsymbol{r'}) = \left\langle \phi(t,\boldsymbol{r})\widetilde{\phi}(s,\boldsymbol{r'})\exp\left(\int \mathrm{d}u\,\mathrm{d}\boldsymbol{R}\,\mathrm{d}u'\,\mathrm{d}\boldsymbol{R'}\,\widetilde{\phi}(u,\boldsymbol{R})\kappa(u,u';\boldsymbol{R}-\boldsymbol{R'})\widetilde{\phi}(u',\boldsymbol{r'})\right)\right\rangle_0 \tag{3.16}$$

where κ is given by eq. (3.7). Formally expanding the exponential and taking the Bargman superselection rule (3.15) into account, only the term of lowest order remains. We thus find

$$R(t,s;\boldsymbol{r},\boldsymbol{r'}) = R_0(t,s;\boldsymbol{r},\boldsymbol{r'}) = \frac{k(t)}{k(s)}\Theta(t-s)\left(4\pi(t-s)\right)^{-d/2}\exp\left(-\frac{1}{4}\frac{(\boldsymbol{r}-\boldsymbol{r'})^2}{(t-s)}\right) \tag{3.17}$$

and we see that R is independent of the noise.

Next, the order-parameter correlation function reads

$$C(t,s;\boldsymbol{r},\boldsymbol{r'}) = \left\langle \phi(t,\boldsymbol{r})\phi(s,\boldsymbol{r'})\exp\left(\int \mathrm{d}u\,\mathrm{d}\boldsymbol{R}\,\mathrm{d}u'\,\mathrm{d}\boldsymbol{R'}\,\widetilde{\phi}(u,\boldsymbol{R})\kappa(u,u';\boldsymbol{R}-\boldsymbol{R'})\widetilde{\phi}(u',\boldsymbol{r'})\right)\right\rangle_0 \tag{3.18}$$

Again expanding the exponential and using eq. (3.15), a single term remains and we readily find

$$\begin{aligned} C(t,s;\boldsymbol{r},\boldsymbol{r'}) &= \int \mathrm{d}\boldsymbol{R}\,\mathrm{d}u\,\mathrm{d}\boldsymbol{R'}\,\mathrm{d}u'\,\kappa(u,u';\boldsymbol{R}-\boldsymbol{R'})R_0^{(4)}(t,s,u,u';\boldsymbol{r},\boldsymbol{r'},\boldsymbol{R},\boldsymbol{R'}) \\ R_0^{(4)}(t,s,u,u';\boldsymbol{r},\boldsymbol{r'},\boldsymbol{R},\boldsymbol{R'}) &:= \left\langle \phi(t,\boldsymbol{r})\phi(s,\boldsymbol{r'})\widetilde{\phi}(u,\boldsymbol{R})\widetilde{\phi}(u',\boldsymbol{R'})\right\rangle_0 \end{aligned} \tag{3.19}$$

where $R_0^{(4)}$ is a noiseless four-point function.

Repeating the same arguments as before, it is an easy task to compute the two-point correlations of response fields. They are

$$\widetilde{C}(t,s;\boldsymbol{r},\boldsymbol{r'}) = 0 \tag{3.20}$$

We emphasize that eqs. (3.16,3.19,3.20) will hold for any theory satisfying the Bargman's superselection rule (3.15). We shall come back to this in section 4.

The four-point function $R_0^{(4)}$ is simple to access for free fields since it factorizes into a product of two-point functions because of Wick's theorem. We have

$$R_0^{(4)}(t,s,u,u';\boldsymbol{r},\boldsymbol{r'},\boldsymbol{R},\boldsymbol{R'}) = R_0(t,u;\boldsymbol{r},\boldsymbol{R})R_0(s,u';\boldsymbol{r'},\boldsymbol{R'}) + R_0(t,u';\boldsymbol{r},\boldsymbol{R'})R_0(s,u;\boldsymbol{r'},\boldsymbol{R}) \tag{3.21}$$

Together with the explicit form of κ, this yields the final result

$$
\begin{aligned}
C(t,s;\boldsymbol{r},\boldsymbol{r}') &= C_{th}(t,s;\boldsymbol{r},\boldsymbol{r}') + C_{pr}(t,s;\boldsymbol{r},\boldsymbol{r}') \\
C_{th}(t,s;\boldsymbol{r},\boldsymbol{r}') &= 2T\int \mathrm{d}u\,\mathrm{d}\boldsymbol{y}\; R_0(t,u;\boldsymbol{r},\boldsymbol{y})R_0(s,u;\boldsymbol{r}',\boldsymbol{y}) \\
C_{pr}(t,s;\boldsymbol{r},\boldsymbol{r}') &= \int \mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{y}'\; R_0(t,0;\boldsymbol{r},\boldsymbol{y})a(\boldsymbol{y}-\boldsymbol{y}')R_0(s,0;\boldsymbol{r}',\boldsymbol{y}')
\end{aligned}
\tag{3.22}
$$

where we separated C in a thermal term C_{th} and an initial term C_{pr}. We clearly see that while the only contributions to C come from the noise, R does not depend on it.

We summarize the results obtained so far as follows:

1. It is satisfying that the well-known result $\widetilde{C} = \langle \widetilde{\phi}\widetilde{\phi}\rangle = 0$ is naturally reproduced, see (3.20).
2. The *independence* eq. (3.17) of the two-time response function of T and of the initial correlations goes beyond the usual scaling arguments as reviewed in section 1. This explains to some extent the success of the existing confirmations of that prediction of local scale invariance.
3. We arrive at an explicit expression for the two-time correlators, which are obtained in terms of a contraction of two response functions. We also see that the earlier result $C = 0$ comes from neglecting both initial and thermal fluctuations.

It is useful to present these results also in momentum space. Using spatial translation invariance, define the Fourier transform of any two-point function $A(t,s;\boldsymbol{x},\boldsymbol{y}) = A(t,s;\boldsymbol{x}-\boldsymbol{y})$ as

$$
\widehat{A}(t,s;\boldsymbol{q}) := \int_{\mathbb{R}^d} \mathrm{d}\boldsymbol{r}\; A(t,s;\boldsymbol{r}) \exp\left(-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}\right) \tag{3.23}
$$

Then eqs. (3.17,3.22) become

$$
\begin{aligned}
\widehat{R}(t,s;\boldsymbol{q}) &= \frac{k(t)}{k(s)} \exp\left(-\boldsymbol{q}^2(t-s)\right)\,\Theta(t-s) \\
\widehat{C}_{th}(t,s;\boldsymbol{q}) &= 2T\int_0^s \mathrm{d}u\; \frac{k(t)k(s)}{k^2(u)} \exp\left(-\boldsymbol{q}^2(t+s-2u)\right) \\
\widehat{C}_{pr}(t,s;\boldsymbol{q}) &= \widehat{R}_0(t,0;\boldsymbol{q})\widehat{a}(\boldsymbol{q})\widehat{R}_0(s,0;-\boldsymbol{q}) = \widehat{a}(\boldsymbol{q})\frac{k(t)k(s)}{k^2(0)}\exp\left(-\boldsymbol{q}^2(t+s)\right)
\end{aligned}
\tag{3.24}
$$

where in the second line, the convention $s < t$ has been used. In particular, if $v(t) = 0$ the two-point correlation function takes an especially simple form $\widehat{C}^0(t,s;\boldsymbol{q})$ where

$$
\widehat{C}^0(t,s;\boldsymbol{q}) = \left[\widehat{a}(\boldsymbol{q}) - \frac{T}{\boldsymbol{q}^2}\right]\exp\left(-\boldsymbol{q}^2(t+s)\right) + \frac{T}{\boldsymbol{q}^2}\exp\left(-\boldsymbol{q}^2(t-s)\right) \tag{3.25}
$$

While both response and correlation functions depend on both t and s and therefore describe an ageing behaviour, there is an *equilibrium regime* $1 \ll t-s \ll s,t$ where we have the following simple expressions

$$
\begin{aligned}
\widehat{R}_{\mathrm{eq}}(t,s;\boldsymbol{q}) &= \exp\left(-\boldsymbol{q}^2(t-s)\right) \\
\widehat{C}^0_{\mathrm{eq}}(t,s;\boldsymbol{q}) &= \frac{T}{\boldsymbol{q}^2}\exp\left(-\boldsymbol{q}^2(t-s)\right)
\end{aligned}
\tag{3.26}
$$

More generally, it is not difficult to show that $\widehat{C}_{\rm eq}(t,s;\boldsymbol{q}) = (T/\boldsymbol{q}^2)\, e^{-\boldsymbol{q}^2(t-s)}\,(1+\mathrm{O}\left((s\boldsymbol{q}^2)^{-1},(t-s)/s\right))$. In any case, we recover the fluctuation-dissipation theorem $T\widehat{R}_{\rm eq}(t,s;\boldsymbol{q}) = \partial \widehat{C}_{\rm eq}(t,s;\boldsymbol{q})/\partial s$ in the equilibrium regime as it should be.

Motivated from studies in spin glasses, it is sometimes attempted to separate correlation and response functions into an equilibrium and an 'ageing' part, viz. $C = C_{\rm eq} + C_{\rm age}$, $R = R_{\rm eq} + R_{\rm age}$. In our case, we would have

$$
\begin{aligned}
\widehat{R}_{\rm age}(t,s;\boldsymbol{q}) &= \left(\frac{k(t)}{k(s)} - 1\right)\exp\left(-\boldsymbol{q}^2(t-s)\right) \\
\widehat{C}^0_{\rm age}(t,s;\boldsymbol{q}) &= \left(\widehat{a}(\boldsymbol{q}) - \frac{T}{\boldsymbol{q}^2}\right)\exp\left(-\boldsymbol{q}^2(t+s)\right)
\end{aligned}
\tag{3.27}
$$

For $v(t)=0$, there is no 'ageing' part in the response function.

From these expressions, we can already extract a few general properties of the ageing process. First, for systems quenched to below their critical temperature $T_c > 0$ and described by a MSR Gaussian action (3.1), it is known from the dynamical renormalization group that the final temperature $T < T_c$ is an irrelevant parameter and furthermore $T \to 0$ under renormalization [9, 11]. Then the long-time dynamics should be driven by the initial fluctuations, in agreement with eq. (3.27) with $T = 0$. On the other hand, for a critical quench $T = T_c$, the situation is different in that both initial and thermal fluctuation may contribute to the long-time dynamics. From eq. (3.27) we expect that the small-$\boldsymbol{q}$ behaviour of the term $\widehat{a}(\boldsymbol{q}) - T_c/\boldsymbol{q}^2$ will determine the long-time dynamics.

4 Consequences of local scale-invariance of noiseless theories

4.1 MSR formulation

We now generalize the results of the previous section to any theory whose noiseless MSR action is Schrödinger-invariant. As we shall stay within the context of classical field theory, the symmetries of the MSR action are the same as for the corresponding Langevin equation (1.8). The formulation of the MSR action, using a non-conserved order parameter described by model A dynamics [53] is almost unchanged with respect to section 3. We have

$$
S[\phi,\widetilde{\phi}] = S_0[\phi,\widetilde{\phi}] + \mathcal{S}[\phi,\widetilde{\phi}] + \sigma[\phi,\widetilde{\phi}]
\tag{4.1}
$$

where $\mathcal{S}[\phi,\widetilde{\phi}]$ and $\sigma[\phi,\widetilde{\phi}]$ are given by eqs. (3.3,3.5) and

$$
S_0[\phi,\widetilde{\phi}] = \int \mathrm{d}\boldsymbol{r}\,\mathrm{d}t\,\widetilde{\phi}\left(\frac{\partial\phi}{\partial t} + \frac{\delta H}{\delta\phi}\right)
\tag{4.2}
$$

We shall assume throughout that *the noiseless action S_0 is Schrödinger-invariant* and that it includes an external time-dependent potential $v = v(t)$. For a local effective potential $\mathcal{H}$, it is known that spatial translation-invariance, dilatation-invariance (or dynamical scaling) and Galilei-invariance are sufficient for having Schrödinger invariance [48]. Therefore, from now on the dynamical exponent $z = 2$.

In order to study the effects of the noise, we first discuss what becomes of the Galilei invariance of the noiseless theory. If the order parameter ϕ and the response field $\widetilde{\phi}$ are quasiprimary, they should transform under a Galilei transformation $t \mapsto t' = t$ and $\boldsymbol{r} \mapsto \boldsymbol{r}' = \boldsymbol{r} - \boldsymbol{v}t$ as (see eq. (2.5))

$$
\begin{aligned}
\phi'(t',\boldsymbol{r}') &= f_{\boldsymbol{v}}(t,\boldsymbol{r})\phi(t,\boldsymbol{r}) \\
\widetilde{\phi}'(t',\boldsymbol{r}') &= f_{\boldsymbol{v}}^{-1}(t,\boldsymbol{r})\widetilde{\phi}(t,\boldsymbol{r})
\end{aligned}
\tag{4.3}
$$

where the companion function $f_{\boldsymbol{v}}$ reads [70]

$$f_{\boldsymbol{v}}(t,\boldsymbol{r}) = \exp\left[\mathcal{M}\boldsymbol{r}\cdot\boldsymbol{v} - \frac{\mathcal{M}}{2}\boldsymbol{v}^2 t\right] \tag{4.4}$$

The noisy contributions to the action transform as

$$\begin{aligned}
\mathcal{S}[\phi',\widetilde{\phi}'] - \mathcal{S}[\phi,\widetilde{\phi}] &= -T\int \mathrm{d}t\,\mathrm{d}\boldsymbol{r}\,\widetilde{\phi}^2(t,\boldsymbol{r})\left(f_{\boldsymbol{v}}^{-2}(t,\boldsymbol{r})-1\right)\\
\sigma[\phi',\widetilde{\phi}'] - \sigma[\phi,\widetilde{\phi}] &= -\frac{1}{2}\int \mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{r}' a(\boldsymbol{r}-\boldsymbol{r}')\widetilde{\phi}(0,\boldsymbol{r})\widetilde{\phi}(0,\boldsymbol{r}')\left(f_{\boldsymbol{v}}^{-1}(0,\boldsymbol{r})f_{\boldsymbol{v}}^{-1}(0,\boldsymbol{r}')-1\right)
\end{aligned} \tag{4.5}$$

Therefore, for fixed temperature and initial conditions, the noise always destroys Galilei invariance.[5] Thus, the only dynamic symmetries of the noisy Langevin equation should be space translations and dilatations (and possibly space rotations).

Consequently, one would merely expect the following scaling forms

$$\begin{aligned}
C(t,s,\boldsymbol{r},\boldsymbol{r}') &= s^{-b}\mathcal{G}_C\left(\frac{t}{s},\frac{(\boldsymbol{r}-\boldsymbol{r}')^2}{t-s}\right)\\
R(t,s;\boldsymbol{r},\boldsymbol{r}') &= s^{-a-1}\mathcal{G}_R\left(\frac{t}{s},\frac{(\boldsymbol{r}-\boldsymbol{r}')^2}{t-s}\right)
\end{aligned} \tag{4.6}$$

where $\mathcal{G}_{C,R}$ are undetermined scaling functions. For free-field theories, one would have $a=b=d/2-1$ by dimensional counting.

4.2 Response and correlation functions for the fluctuating theory

The form of the scaling functions $\mathcal{G}_{R,C}$ will now be determined through a generalization of the expansions carried out in section 3. The main results will be eqs. (4.8) and (4.12), (4.16), respectively.

4.2.1 The two-point response function

Consider a system which initially is at thermal equilibrium with temperature T_i (which fixes the initial correlator $a(\boldsymbol{r})$) and quench it at time $t=0$ to the final temperature $T=T_f$. The response function is still given by eq. (3.16) and the expansion of the exponential goes through as before. Because of the Bargman superselection rule (3.15) which holds because of the assumed Galilei invariance of S_0 only the lowest term survives and we obtain

$$R(t,s;\boldsymbol{r},\boldsymbol{r}') = R_0(t,s;\boldsymbol{r},\boldsymbol{r}') \tag{4.7}$$

The expression of R_0 has been derived earlier and using the gauge transform (2.4) we recover exactly the same result as in eq. (2.12), namely

$$R\left(t,s;\boldsymbol{r},\boldsymbol{r}'\right) = \delta_{x,\widetilde{x}}\,r_0\,\Theta(t-s)\,(t-s)^{-x}\,\frac{k(t)}{k(s)}\exp\left(-\frac{\mathcal{M}}{2}\frac{(\boldsymbol{r}-\boldsymbol{r}')^2}{(t-s)}\right) \tag{4.8}$$

We stress that, given only Galilei- and scale-invariance of the noiseless theory, this result should hold for any initial and final temperature T_i and T_f. At this stage, nothing has yet been said on the time-dependence of $v(t)$.

[5] The breaking of Galilei-invariance through thermal noise can be visualized as follows: consider a system in contact with a thermal bath at constant and uniform temperature $T>0$. If the system moves with respect to the bath with a constant speed $\boldsymbol{v}$, the apparent temperature measured in the system will depend on the angle between the direction of measurement and $\boldsymbol{v}$.

4.2.2 The two-point correlation function

The two-point correlation function is found from eq. (3.19). Again, the arguments of section 3 go through and we find, using the explicit expression (3.7) for the kernel κ

$$C(t,s;\boldsymbol{r},\boldsymbol{r}') = T\int \mathrm{d}u\,\mathrm{d}\boldsymbol{R}\,R_0^{(3)}(t,s,u;\boldsymbol{r},\boldsymbol{r}',\boldsymbol{R}) + \frac{1}{2}\int \mathrm{d}\boldsymbol{R}\,\mathrm{d}\boldsymbol{R}'\,a(\boldsymbol{R}-\boldsymbol{R}')R_0^{(4)}(t,s,0,0;\boldsymbol{r},\boldsymbol{r}',\boldsymbol{R},\boldsymbol{R}') \quad (4.9)$$

where $R_0^{(3)}$ is the following three-point response function

$$R_0^{(3)}(t,s,u;\boldsymbol{r},\boldsymbol{r}',\boldsymbol{R}) := \left\langle \phi(t;\boldsymbol{r})\phi(s;\boldsymbol{r})\widetilde{\phi}(u;\boldsymbol{R})^2\right\rangle_0 \quad (4.10)$$

and $R_0^{(4)}$ was already defined in eq. (3.19). This central result will be the basis for all what follows. Consequently, the calculation of C requires the computation of the *noiseless* three- and four-point functions $R_0^{(3)}$ and $R_0^{(4)}$. This cannot entirely be done, since a general expression for $R_0^{(4)}$ is not yet available. Of course, one might hope that through an extension of the Schrödinger algebra $\mathfrak{sch}_d$ to some infinite-dimensional Lie algebra techniques analogous to $2D$ conformal invariance [3] might become applicable but the formulation of just such an extension is an open problem.

Here, we shall restrict to the case of vanishing initial correlations, that is

$$a(\boldsymbol{R}) = a_0\delta(\boldsymbol{R}) \quad (4.11)$$

which corresponds to an infinite initial temperature $T_i=\infty$ and where a_0 is a normalization constant. In fact, from renormalization group arguments the long-time behaviour of any system which is prepared in the high-temperature or paramagnetic phase should be described by this initial condition [9, 11]. Then

$$\begin{aligned} C(t,s) &= T\int \mathrm{d}u\,\mathrm{d}\boldsymbol{R}\,R_0^{(3)}(t,s,u;\boldsymbol{R}) + \frac{a_0}{2}\int \mathrm{d}\boldsymbol{R}\,R_0^{(3)}(t,s,0;\boldsymbol{R}) && (4.12)\\ R_0^{(3)}(t,s,u;\boldsymbol{r}) &:= R_0^{(3)}(t,s,u;\boldsymbol{y},\boldsymbol{y},\boldsymbol{r}+\boldsymbol{y}) = \left\langle \phi(t;\boldsymbol{y})\phi(s;\boldsymbol{y})\widetilde{\phi}(u;\boldsymbol{r}+\boldsymbol{y})^2\right\rangle_0 && (4.13)\end{aligned}$$

Here, the field $\widetilde{\phi}^2$ is a composite field with mass $-2\mathcal{M}$ and scaling dimension $2\widetilde{x}_2$. Only for free fields, one has $\widetilde{x}_2=\widetilde{x}$.

Now, the three-point function of a Schrödinger-invariant theory with $v(t)=0$ is well-known since a long time [43, 46]. Denoting by $\mathcal{R}$ the response function in the case where $v(t)=0$, we have

$$\begin{aligned} \mathcal{R}_0^{(3)}(t,s,u;\boldsymbol{r}) &= \mathcal{R}_0^{(3)}(t,s,u)\exp\left[-\frac{\mathcal{M}}{2}\frac{t+s-2u}{(s-u)(t-u)}\boldsymbol{r}^2\right]\Psi\left(\frac{t-s}{(t-u)(s-u)}\boldsymbol{r}^2\right)\\ \mathcal{R}_0^{(3)}(t,s,u) &= \Theta(t-u)\Theta(s-u)\,(t-u)^{-\widetilde{x}_2}\,(s-u)^{-\widetilde{x}_2}\,(t-s)^{-x+\widetilde{x}_2} && (4.14)\end{aligned}$$

where Ψ is an arbitrary scaling function and $\mathcal{M}$ is the 'mass' of the field ϕ. This result is brought to the case at hand through the gauge transformation (2.4) and we find

$$R_0^{(3)}(t,s,u;\boldsymbol{r}) = \frac{k(t)k(s)}{k^2(u)}\mathcal{R}_0^{(3)}(t,s,u;\boldsymbol{r}) \quad (4.15)$$

Combining (4.12) and (4.15) we obtain, for the first time, a generic prediction for the form of the two-point correlation function. It is remarkable that under the condition (4.11) only the noiseless three-point response functions are required in order to predict *any* two-time autocorrelator.

It is useful to write down the autocorrelation function in the form $C(t,s) = C_{th}(t,s) + C_{pr}(t,s)$. Here the thermal part C_{th} and the preparation part C_{pr} are given by

$$
\begin{aligned}
C_{th}(t,s) &= T s^{d/2+1-x-\widetilde{x}_2} \left(\frac{t}{s}-1\right)^{\widetilde{x}_2-x-d/2} \int_0^1 \mathrm{d}\theta\, \frac{k(t)k(s)}{k^2(s\theta)} \left[\left(\frac{t}{s}-\theta\right)(1-\theta)\right]^{d/2-\widetilde{x}_2} \Phi\left(\frac{t/s+1-2\theta}{t/s-1}\right) \\
C_{pr}(t,s) &= \frac{a_0}{2}\frac{k(t)k(s)}{k^2(0)} s^{d/2-\widetilde{x}_2-x} \left(\frac{t}{s}\right)^{d/2-\widetilde{x}} \left(\frac{t}{s}-1\right)^{\widetilde{x}_2-x-d/2} \Phi\left(\frac{t/s+1}{t/s-1}\right) \\
\Phi(w) &:= \int \mathrm{d}\boldsymbol{R}\, \exp\left[-\frac{\mathcal{M}w}{2}\boldsymbol{R}^2\right] \Psi\left(\boldsymbol{R}^2\right)
\end{aligned}
\tag{4.16}
$$

and we have explicitly used $s < t$.

As they stand, the above expressions for $C(t,s)$ do not yet necessarily describe a dynamical scaling behaviour, since the form of $k(t)$ is still completely general. We now assume in addition that we are dealing with a system with dynamical scaling. In order to reproduce the usual phenomenology of ageing systems, we must have, at least for sufficiently large times (see section 2)

$$
k(t) \simeq k_0 t^{\digamma}
\tag{4.17}
$$

Comparing the general form of $R(t,s)$ as given in eq. (4.8) with the phenomenologically expected scaling eqs. (1.2,1.3), we read off

$$
x = 1 + a \;\;,\;\; \digamma = 1 + a - \frac{\lambda_R}{2}
\tag{4.18}
$$

and in particular, the scaling function (1.7) is recovered. On the other hand, for $C(t,s)$ we find the following scaling form, written down separately for the thermal and the initial term, where $y = t/s \geq 1$ is the scaling variable

$$
\begin{aligned}
C_{th}(t,s) &= T s^{-b_{th}} f_C^{\rm th}(y) \\
C_{pr}(t,s) &= s^{-b_{pr}} f_C^{\rm pr}(y)
\end{aligned}
\tag{4.19}
$$

where the scaling functions $f_C^{\rm pr}$ and $f_C^{\rm th}$ are given by

$$
\begin{aligned}
f_C^{\rm th}(y) &= y^{\digamma} (y-1)^{\widetilde{x}_2-x-d/2} \int_0^1 \mathrm{d}\theta\, \theta^{-2\digamma} \left[(y-\theta)(1-\theta)\right]^{d/2-\widetilde{x}_2} \Phi\left(\frac{y+1-2\theta}{y-1}\right) \\
f_C^{\rm pr}(y) &= \frac{a_0}{2} y^{d/2-\widetilde{x}_2+\digamma} (y-1)^{\widetilde{x}_2-x-d/2} \Phi\left(\frac{y+1}{y-1}\right)
\end{aligned}
\tag{4.20}
$$

with the non-equilibrium exponents

$$
\begin{aligned}
b_{th} &= x + \widetilde{x}_2 - 1 - d/2 \\
b_{pr} &= x + \widetilde{x}_2 - 2\digamma - d/2
\end{aligned}
\tag{4.21}
$$

Provided that $\Phi(1)$ is finite, the asymptotic behaviour of the scaling functions for y large can be worked out. We expect $f_C(y) \sim y^{-\lambda_C/2}$ and find

$$
\lambda_C^{th} = \lambda_C^{pr} = 2(x - \digamma)
\tag{4.22}
$$

Therefore, comparing this with (4.18), we have shown:

For any system with an infinite-temperature initial state (4.11), quenched to a temperature $T < T_c$ and whose noiseless part is locally scale-invariant with $z = 2$, one has

$$\lambda_C = \lambda_R. \tag{4.23}$$

For non-equilibrium critical dynamics (that is $T = T_c$) the same conclusion can be drawn if after renormalization one still has $z = 2$. While eq. (4.23) certainly agrees with the evidence available from models studied either analytically or numerically, we are not aware of any other general proof of this equality between the autocorrelation and autoresponse exponents for a fully disordered initial state.

In sections 5-7, we shall present extensive tests of the prediction (4.19,4.20,4.21) for C and (4.8,4.18) for R, respectively. The main hypothesis going into it is the requirement of Galilei-invariance of the noiseless theory, while the other conditions appear to be habitually admitted in the description of physical ageing.

4.2.3 Autocorrelation function in phase-ordering kinetics

In order to understand the result (4.19) for $C(t,s)$ better, we now study the two contributions separately. First, we consider the 'preparation' part C_{pr}. This term is expected to describe the late-time behaviour of a system quenched to a temperature $T < T_c$. Indeed, renormalization-group arguments show [9] that in this case T is an irrelevant variable and is renormalized towards zero. Then the thermal contribution C_{th} vanishes. Therefore, the non-equilibrium exponent $b = b_{pr}$ is read off from eq. (4.21). In addition, we know that $b = 0$ in the low-temperature phase. This implies

$$\widetilde{x}_2 - x = \frac{d}{2} - \lambda_C \leq 0 \tag{4.24}$$

because of a well-known inequality [93]. Only for free fields, one has $\lambda_C = d/2$, otherwise $\widetilde{x}_2$ is a new nontrivial exponent. In the scaling limit, we thus have $C(t,s) = f_C(t/s)$ where

$$f_C(y) = \frac{a_0}{2} y^{\lambda_C/2} (y-1)^{-\lambda_C}\, \Phi\left(\frac{y+1}{y-1}\right) \tag{4.25}$$

The form of $f_C(y)$ still depends on the unknown function $\Phi(w)$ which in turn depends on $\Psi(\rho)$, see (4.16). We attempt to fix its form and reconsider the noiseless response function $R_0^{(3)}(t,s,0;\boldsymbol{r})$ which describes a response of the autocorrelation $C(t,s) = \langle\phi(t)\phi(s)\rangle$. It appears to be a reasonable requirement that there should be no singularity in $R_0^{(3)}$ when $t = s$. Using the explicit form eqs. (4.14,4.15) this leads to the following limit behaviour

$$\Psi(\rho) \simeq \Psi_0 \rho^{\lambda_C - d/2} \;\; ; \;\; \rho \to 0 \tag{4.26}$$

where Ψ_0 is a constant. If this leading term should be still accurate for larger values of ρ, the following expression for the scaling function $\Phi(w)$ is found

$$\Phi(w) \approx \Psi_0 S_d \frac{\Gamma(\lambda_C)}{2}\left(\frac{2}{\mathcal{M}}\right)^{\lambda_C} \cdot w^{-\lambda_C} =: \Phi_0 \cdot w^{-\lambda_C} \tag{4.27}$$

where S_d is the surface of the unit sphere in d dimensions. Eq. (4.27) should hold in the $w \to \infty$ limit. Provided this from is still valid for all w, we would obtain the following simplified form, with $y = t/s$

$$C(t,s) \approx \frac{a_0\Phi_0}{2}\left(\frac{(y+1)^2}{y}\right)^{-\lambda_C/2} = M_{\rm eq}^2 \left(\frac{(y+1)^2}{4y}\right)^{-\lambda_C/2} \;\; ; \;\; T = 0 \tag{4.28}$$

where we also related the leading constant to the squared equilibrium magnetization $M_{\rm eq}^2$, in order to recover the usual scaling form $C(t,s) = M_{\rm eq}^2 f_C(t/s)$ with $f_C(1) = 1$, see [39].

4.2.4 Autocorrelation function for critical dynamics

Second, let us turn to the thermal part. It should dominate the autocorrelation for quenches to the critical point $T = T_c$, given the initial condition (4.11) and under the assumption that $z = 2$ even at criticality. From eq. (4.21), the exponent $b = b_{th}$ can be read off and we have

$$\widetilde{x}_2 = b - a + \frac{d}{2} \tag{4.29}$$

Under the stated conditions, the preparation term drops out at large times and we find

$$\begin{aligned} C(t,s) &= T_c s^{-b} f_C(t/s) \\ f_C(y) &= y^F (y-1)^{b-2a-1} \int_0^1 \mathrm{d}\theta\, \theta^{-2F} \left[(y-\theta)(1-\theta)\right]^{a-b} \Phi\left(\frac{y+1-2\theta}{y-1}\right) \end{aligned} \tag{4.30}$$

At criticality, one expects the following relationship between the nonequilibrium exponents a and b

$$a = b = \frac{2\beta}{\nu z} = \frac{d-2+\eta}{z} \tag{4.31}$$

where β, ν, η are well-known equilibrium critical exponents (see section 1). To understand eq. (4.31), recall that $C(s,s) \sim s^{-b}$. On the other hand, from the space-time scaling $|\boldsymbol{r}|^z \sim t$, one expects the equilibrium correlator to decay as $C_{\rm eq} \sim |\boldsymbol{r}|^{-bz}$ and the second equality in eq. (4.31) follows. Finally, $a = b$ is a necessary condition for having a non-vanishing limit fluctuation-dissipation ratio X_∞.

Now, if we let $a = b$ in (4.30), we find

$$f_C(y) = y^F (y-1)^{-1-a} \int_0^1 \mathrm{d}\theta\, \theta^{-2F} \Phi\left(\frac{y+1-2\theta}{y-1}\right) \tag{4.32}$$

which is the most general form compatible with the standard phenomenological constraints. An approximate form of the scaling function $\Phi(w)$ may be obtained from the requirement that the three-point response function $\mathcal{R}_0^{(3)}(t,s,u;\boldsymbol{r})$ is non-singular for $t = s$. We now have $\widetilde{x}_2 = d/2$ and find $\Psi(\rho) \simeq \Psi_{0,c}\, \rho^{x-d/2}$ as $\rho \to 0$. This leads to, for $w \to \infty$

$$\Phi(w) \approx \Phi_{0,c}\, w^{-1-a} \tag{4.33}$$

where $\Phi_{0,c}$ is a constant. If in addition we may use this form also for finite values of w, we would obtain the simplified form

$$f_C(y) \approx \Phi_{0,c}\, y^{1+a-\lambda_C/2} \int_0^1 \mathrm{d}\theta\, \theta^{\lambda_C-2-2a} (y+1-2\theta)^{-1-a} \quad ; \quad T = T_c \tag{4.34}$$

Summarizing, the phenomenological comparison of the autocorrelation function, as predicted by Schrödinger-invariance, and assuming a totally disordered initial state, with simulational or experimental data will be based on eqs. (4.25) and (4.32) for quenches to $T < T_c$ and $T = T_c$, respectively. In full generality this will allow to obtain information on the scaling function $\Phi(w)$. If in addition the heuristic idea of the absence of singularities at $t = s$ in the three-point response function $R_0^{(3)}$ should be sufficient to fix the form of this response function, the simplified forms (4.28,4.34) apply and the scaling function $f_C(y)$ is completely specified in terms of the exponents a and λ_C.

5 Tests of local scale-invariance in exactly solvable models

In this and the next two sections we describe phenomenological tests of the predictions of Schrödinger-invariance, that is local scale-invariance with $z = 2$, which were derived in section 4 in concrete physical models of ageing behaviour.

5.1 Kinetic spherical model

The kinetic spherical model is often formulated in a field-theoretic fashion as the $n \to \infty$ limit of the $\mathrm{O}(n)$-symmetric vector model. For our purposes, it is more useful to start directly from a lattice system and to take the continuum limit later, following [22, 38, 95, 77, 78, 75].

Consider a hypercubic lattice with $\mathcal{N}$ sites. At each site $\boldsymbol{r}$ there is a real time-dependent variable $\phi(t, \boldsymbol{r})$ such that the mean spherical constraint

$$\left\langle \sum_{\boldsymbol{r}} \phi(t, \boldsymbol{r})^2 \right\rangle = \mathcal{N} \tag{5.1}$$

holds. The Hamiltonian is $\mathcal{H} = -\sum_{<\boldsymbol{r},\boldsymbol{r}'>} \phi(t, \boldsymbol{r})\phi(t, \boldsymbol{r}')$ where the sum extends over pairs of nearest neighbour sites. The (non-conserved) dynamics is given in terms of a Langevin equation of the type eq. (1.8)

$$\begin{aligned} \frac{\partial \phi(t, \boldsymbol{r})}{\partial t} &= \sum_{\boldsymbol{n}(\boldsymbol{r})} \phi(t, \boldsymbol{n}) - (2d + v(t))\phi(t, \boldsymbol{r}) + \eta(t, \boldsymbol{r}) & (5.2) \\ &\simeq \Delta\phi(t, \boldsymbol{r}) - v(t)\phi(t, \boldsymbol{r}) + \eta(t, \boldsymbol{r}) & (5.3) \end{aligned}$$

where $\boldsymbol{n}(\boldsymbol{r})$ runs over the nearest neighbours of the site $\boldsymbol{r}$. In the second line, a formal continuum limit was taken (for simplicity, all rescalings with powers of the lattice constant a were suppressed). Finally, $\eta(t, \boldsymbol{r})$ is the usual Gaussian noise (see section 1) and the Lagrange multiplier $v(t)$ is fixed such that the mean spherical constraint is satisfied.[6] Therefore, the kinetic spherical model perfectly fits into the context of a Schrödinger equation in a time-dependent potential $v = v(t)$ discussed in section 3. We therefore expect that the free-field predictions eqs. (3.17,3.22) will be fully confirmed.

In order to see this explicitly, we recall the elements of the exact solution of the Langevin equation (5.2). If we set

$$g(t) = \exp\left(2\int_0^t \mathrm{d}u\, v(u)\right) \tag{5.4}$$

it can be shown [22, 38, 77] that $g(t)$ is the unique solution of the Volterra integral equation

$$g(t) = A(t) + 2T\int_0^t \mathrm{d}t'\, f(t - t')g(t') \tag{5.5}$$

where $g(0) = 1$ and

$$\begin{aligned} f(t) &= \Theta(t)\left(e^{-4t} I_0(4t)\right)^d \\ A(t) &= \frac{1}{(2\pi)^d}\int_{\mathcal{B}} \mathrm{d}\boldsymbol{q} \sum_{\boldsymbol{r}} a(\boldsymbol{r}) e^{-2\omega(\boldsymbol{q})t - \mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}} & (5.6) \end{aligned}$$

[6] A careful study shows that provided the limit $\mathcal{N} \to \infty$ is taken *before* any long-time limit, the mean spherical constraint (5.1) and a full, non-averaged, spherical constraint lead to the same results [29].

and $\omega(\boldsymbol{q}) = 2\sum_{i=1}^{d}(1-\cos q_i)$ is the lattice dispersion relation, $\mathcal{B}$ the Brillouin zone while I_0 is a modified Bessel function. We stress that because of eq. (5.5), $g(t)$ does *not* depend on the order parameter $\phi(t,\boldsymbol{r})$.

We now show that the prediction (3.17,3.22) for the two-point functions can be fully reproduced. We begin with the response function. In the spherical model, it is exactly given by [38, 77]

$$\begin{aligned} R(t,s;\boldsymbol{r},\boldsymbol{r}') &= \left.\frac{\delta\langle\phi(t,\boldsymbol{r})\rangle}{\delta h(s,\boldsymbol{r}')}\right|_{h=0} \\ &= \left[\prod_{i=1}^{d} e^{-2(t-s)} I_{\boldsymbol{r}_i-\boldsymbol{r}_i'}\left(2(t-s)\right)\right]\sqrt{\frac{g(s)}{g(t)}} \\ &\simeq (4\pi(t-s))^{-d/2}\exp\left(-\frac{(\boldsymbol{r}-\boldsymbol{r}')^2}{4(t-s)}\right)\exp\left(-\int_s^t \mathrm{d}u\, v(u)\right)\Theta(t-s) \end{aligned} \tag{5.7}$$

where in the second line the limit $t-s \gg 1$ was taken and I_r is a modified Bessel function. In particular, this reproduces the known autoresponse function [38]

$$R(t,s) = R(t,s;\boldsymbol{r},\boldsymbol{r}) = f((t-s)/2)\sqrt{g(s)/g(t)} \tag{5.8}$$

This is in exact agreement with eq. (3.17) and we identify the mass $\mathcal{M} = 1/2$.

We now turn to the correlator but the sake of brevity merely deal with the autocorrelator explicitly. In the spherical model one has (with $t > s$) [38, 77, 75]

$$C(t,s) = \langle\phi(t,\boldsymbol{r})\phi(s,\boldsymbol{r})\rangle = \frac{1}{\sqrt{g(t)g(s)}}\left[A\left(\frac{t+s}{2}\right) + 2T\int_0^s \mathrm{d}u\, f\left(\frac{t+s}{2}-u\right)g(u)\right] \tag{5.9}$$

In order to show how to recover this explicitly from eq. (3.22), we write again $C(t,s) =: C_{pr} + C_{th}$ and discuss the two terms separately. The first one is, where we use the explicit form (5.7) of $R = R_0$

$$\begin{aligned} C_{pr} &= \int_{\mathbb{R}^{2d}} \mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{y}'\; (4\pi t\, 4\pi s)^{-d/2}\sqrt{\frac{g(0)}{g(t)}\frac{g(0)}{g(s)}}\exp\left[-\frac{(\boldsymbol{r}-\boldsymbol{y})^2}{4t} - \frac{(\boldsymbol{r}-\boldsymbol{y}')^2}{4s}\right] \\ &= \frac{(2\pi)^{-d/2}}{\sqrt{g(t)g(s)}}\int_{\mathbb{R}^{2d}} \mathrm{d}\boldsymbol{q}\mathrm{d}\boldsymbol{q}'\; e^{-\boldsymbol{q}^2 t-\boldsymbol{q}'^2 s+\mathrm{i}(\boldsymbol{q}+\boldsymbol{q}')\cdot\boldsymbol{r}} \cdot J_d \end{aligned} \tag{5.10}$$

where

$$J_d := \int_{\mathbb{R}^{2d}} \mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{y}'\; a(\boldsymbol{y}-\boldsymbol{y}')\, e^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{y}-\mathrm{i}\boldsymbol{q}'\cdot\boldsymbol{y}'} \tag{5.11}$$

and we used $g(0) = 1$. In order to calculate J_d, we set $\boldsymbol{\zeta} = \boldsymbol{y}-\boldsymbol{y}'$, $\boldsymbol{\zeta}' = \boldsymbol{y}+\boldsymbol{y}'$. Then the Jacobian $|\partial(\boldsymbol{y},\boldsymbol{y}')/\partial(\boldsymbol{\zeta},\boldsymbol{\zeta}')| = 2^{-d}$ and we have

$$\begin{aligned} J_d &= 2^{-d}\int_{\mathbb{R}^{2d}} \mathrm{d}\boldsymbol{\zeta}\mathrm{d}\boldsymbol{\zeta}'\; a(\boldsymbol{\zeta})\, e^{-\mathrm{i}\frac{\boldsymbol{\zeta}'}{2}\cdot(\boldsymbol{q}+\boldsymbol{q}')} e^{-\mathrm{i}\frac{\boldsymbol{\zeta}}{2}\cdot(\boldsymbol{q}-\boldsymbol{q}')} \\ &= (2\pi)^d\delta(\boldsymbol{q}+\boldsymbol{q}')\int_{\mathbb{R}^d} \mathrm{d}\boldsymbol{\zeta}\; a(\boldsymbol{\zeta}) e^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{\zeta}} \end{aligned} \tag{5.12}$$

We finally obtain

$$C_{pr} = \frac{1}{\sqrt{g(t)g(s)}}\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \mathrm{d}\boldsymbol{q}\; e^{-\boldsymbol{q}^2(t+s)}\int_{\mathbb{R}^d} \mathrm{d}\boldsymbol{\zeta}\; a(\boldsymbol{\zeta})\, e^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{\zeta}} \tag{5.13}$$

When the support of $a(\boldsymbol{\zeta})$ is restricted to the hypercubic lattice, we therefore have indeed for long times $t+s \gg 1$

$$C_{pr} \simeq \frac{A\left((t+s)/2\right)}{\sqrt{g(t)g(s)}} \tag{5.14}$$

in agreement with the first term in (5.9). The second term in (3.22) is analyzed as follows, for $t > s$

$$\begin{aligned}
C_{th} &= 2T\int_0^\infty \mathrm{d}u \int_{\mathbb{R}^d} \mathrm{d}\boldsymbol{y}\; \left(4\pi(t-u)\,4\pi(s-u)\right)^{-d/2} \sqrt{\frac{g(u)}{g(t)}\frac{g(u)}{g(s)}} \\
&\quad \times \exp\left[-\frac{(\boldsymbol{r}-\boldsymbol{y})^2}{4(t-u)} - \frac{(\boldsymbol{r}-\boldsymbol{y})^2}{4(s-u)}\right] \Theta(t-u)\Theta(s-u) \\
&= 2T\int_0^s \mathrm{d}u \int_{\mathbb{R}^{3d}} \mathrm{d}\boldsymbol{q}\mathrm{d}\boldsymbol{q}'\mathrm{d}\boldsymbol{y}\; e^{-\boldsymbol{q}^2 t - \boldsymbol{q}'^2 s + (\boldsymbol{q}^2+\boldsymbol{q}'^2)u + \mathrm{i}(\boldsymbol{q}+\boldsymbol{q}')\cdot\boldsymbol{r}}\, e^{-\mathrm{i}(\boldsymbol{q}+\boldsymbol{q}')\cdot\boldsymbol{y}} \frac{g(u)}{\sqrt{g(t)g(s)}} \\
&= \frac{2T}{\sqrt{g(t)g(s)}} \int_0^s \mathrm{d}u\, g(u)\, (2\pi)^{-d} \int_{\mathbb{R}^d} \mathrm{d}\boldsymbol{q}\; e^{-2\boldsymbol{q}^2((t+s)/2-u)} \\
&\simeq \frac{2T}{\sqrt{g(t)g(s)}} \int_0^s \mathrm{d}u\, g(u) f\left(\frac{t+s}{2} - u\right)
\end{aligned} \tag{5.15}$$

where the last line holds for sufficiently large arguments of the function f. Taken together with C_{pr}, the expected agreement between the general result eq. (3.22) and the exact expression (5.9) for the spherical model is thus recovered. The full space-time correlator can be checked similarly.

It is interesting to note that for the confirmation of R, see (5.7), we need the condition $t-s \gg 1$, while for the confirmation of C, we also need $s \gg 1$ (which implies $t+s \gg 1$).

So far, we have not yet used the explicit form of $g(t)$ which follows from eq. (5.5). Indeed, it is well-known that for long times, one has [38, 77, 46]

$$g(t) \sim t^{-2F} \tag{5.16}$$

For the fully disordered initial conditions (4.11), the exponent F takes the following values: (i) for $T < T_c$, one has $F = d/4$ and (ii) for $T = T_c$, one has $F = 1 - d/4$ if $2 < d < 4$ and $F = 0$ if $d > 4$. This is exactly the form expected from a potential of the form $v(t) \simeq F/t$, see eq. (2.14).

It is instructive to compare also the explicit result for the two-point functions with the expectations coming from local scale-invariance. For the two-time response function, this confirmation has already been carried out for both the autoresponse function $R(t,s)$ as well as the space-time response $R(t,s;\boldsymbol{r})$ [45, 46, 77] and need not be repeated here. Therefore we concentrate on the two-time autocorrelation function $C(t,s)$. First, we consider the case $T < T_c$ where from the exact solution it is well-known that [38, 69]

$$C(t,s) = M_{\mathrm{eq}}^2 f_C(t/s) \;\;,\;\; f_C(y) = \left(\frac{(y+1)^2}{4y}\right)^{-d/4} \tag{5.17}$$

and we read off from the $y \to \infty$ limit the exponent $\lambda_C = d/2$. Clearly, this exact result is in full agreement with the prediction (4.28) of local scale invariance. Second, we consider the case $T = T_c$. Then the exact solution gives [38]

$$C(t,s) = (4\pi)^{-d/2} T_c s^{-d/2+1} f_C(t/s) \;\;,\;\; f_C(y) = \begin{cases} \frac{4}{d-2}(y-1)^{-d/2+1} y^{1-d/4} (y+1)^{-1} & ;\text{ if } 2 < d < 4 \\[2ex] \frac{2}{d-2}\left[(y-1)^{-d/2+1} - (y+1)^{-d/2+1}\right] & ;\text{ if } d > 4 \end{cases} \tag{5.18}$$

From this, we read off the exponents $a = b = d/2 - 1$ for all $d > 2$ and furthermore $\lambda_C = 3d/2 - 2$ for $2 < d < 4$ and $\lambda_C = d$ for $d > 4$, respectively. Inserting the exponent values into (4.34) it is straightforward to show that eq. (5.18) is indeed reproduced.

In particular, we see that in the spherical model the asymptotic forms eqs. (4.27,4.33), respectively, are indeed exact as should be expected for a free-field theory.

In conclusion, the exact results of the kinetic spherical model for both two-time correlation and response functions are in full agreement with the predictions of Schrödinger invariance.

5.2 XY model in spin-wave approximation

Another system which can be exactly analysed is the kinetic XY model in spin-wave approximation. As we shall see, it provides an instructive example on the correct identification of the quasi-primary scaling fields in a given model.

5.2.1 Formulation and observables

The XY model describes the interaction between planar spin variables

$$\boldsymbol{S}(\boldsymbol{r}) = \cos(\phi(\boldsymbol{r}))\boldsymbol{e}_1 + \sin(\phi(\boldsymbol{r}))\boldsymbol{e}_2 = \begin{pmatrix} \cos\phi(\boldsymbol{r}) \\ \sin\phi(\boldsymbol{r}) \end{pmatrix} \tag{5.19}$$

which are attached to the sites $\boldsymbol{r}$ of a d-dimensional hypercubic lattice and $\phi(\boldsymbol{r})$ is the phase. The Hamiltonian is

$$\mathcal{H}[\phi] = -\sum_{\langle \boldsymbol{r},\boldsymbol{r}'\rangle} \boldsymbol{S}(\boldsymbol{r})\cdot\boldsymbol{S}(\boldsymbol{r}') = -\sum_{\langle \boldsymbol{r},\boldsymbol{r}'\rangle} \cos\left(\phi(\boldsymbol{r}) - \phi(\boldsymbol{r}')\right) \tag{5.20}$$

where the coupling constant J has been set to unity and the sum runs over nearest neighbours. The relaxational dynamics is assumed to be described by a Langevin equation. We prepare the system initially a temperature T_i and quench it at time $t = 0$ to the final temperature $T = T_f$, so that the angular variable obeys [9]

$$\frac{\partial\phi(t,\boldsymbol{r})}{\partial t} = -\frac{\delta\mathcal{H}[\phi]}{\delta\phi(t,\boldsymbol{r})} + \eta(t,\boldsymbol{r}). \tag{5.21}$$

where η represents an uncorrelated Gaussian noise with zero mean and variance

$$\langle\eta(t,\boldsymbol{r})\eta(t',\boldsymbol{r}')\rangle = 2T_f\delta(t-t')\delta(\boldsymbol{r}-\boldsymbol{r}') \tag{5.22}$$

Here, we shall exclusively study the coarsening dynamics in the low-temperature regime, that is

$$T_i, T_f \ll T_c(d) \tag{5.23}$$

where $T_c(d)$ is the critical temperature of the XY model in d dimensions (if $d = 2$, $T_c(2) = T_{\rm KT}$ is the Kosterlitz-Thouless temperature of the transition). Then the so-called spin-wave approximation [84, 4] can be used which amounts to expand $\mathcal{H}$ in powers of $\phi(\boldsymbol{r}) - \phi(\boldsymbol{r}')$. Shifting the energy by a constant, the Hamiltonian reads to lowest order

$$\mathcal{H}[\phi] = \frac{1}{2}\int \mathrm{d}\boldsymbol{r}\ \left(\boldsymbol{\nabla}\phi(\boldsymbol{r})\right)^2 \tag{5.24}$$

In writing this, we have implicitly absorbed the spin-wave stiffness [61, 84] into a redefinition of the temperatures. In $2D$, it is known that below $T_{\rm KT}$, any vortices present will be tightly bound and for distances larger than the characteristic pair size, the XY model renormalizes to the Hamiltonian (5.24) [84].[7]

We are interested in the properties of the two-point functions. It appears natural to define two-time correlation and linear response functions in terms of the magnetic variables

$$\begin{aligned}\Gamma(t,s;\boldsymbol{r},\boldsymbol{r}') &:= \langle \boldsymbol{S}(t,\boldsymbol{r})\cdot\boldsymbol{S}(s,\boldsymbol{r}')\rangle = \langle\cos\left(\phi(t,\boldsymbol{r})-\phi(s,\boldsymbol{r}')\right)\rangle \\ \rho(t,s;\boldsymbol{r},\boldsymbol{r}') &:= \lim_{\boldsymbol{h}\to\boldsymbol{0}} \frac{\partial\langle\boldsymbol{S}(t,\boldsymbol{r})\rangle}{\partial\boldsymbol{h}(s,\boldsymbol{r}')}\end{aligned} \tag{5.25}$$

where the response is found by adding a term $\delta\mathcal{H}_{\rm mag} = \sum_{\boldsymbol{r}} \boldsymbol{h}\cdot\boldsymbol{S}$ to $\mathcal{H}$. Alternatively, one may also consider the analogous quantities defined for the angular variables

$$\begin{aligned}C(t,s;\boldsymbol{r},\boldsymbol{r}') &:= \langle\phi(t,\boldsymbol{r})\phi(s,\boldsymbol{r}')\rangle \\ R(t,s;\boldsymbol{r},\boldsymbol{r}') &:= \lim_{h\to\boldsymbol{0}} \frac{\partial\langle\phi(t,\boldsymbol{r})\rangle}{\partial h(s,\boldsymbol{r}')}\end{aligned} \tag{5.26}$$

(where a perturbation $\delta\mathcal{H}_{\rm ang} = \sum_{\boldsymbol{r}} h\phi$ should have been added).

In order that the spin-wave approximation be applicable, we must start from an (almost) ordered initial state of the system. Therefore, we require the following initial value for the magnetic correlator, which reads in Fourier space

$$\widehat{a}(\boldsymbol{q}) = \widehat{C}(0,0;\boldsymbol{q}) = \frac{2\pi\eta(T_i)}{q^2} = \frac{T_i}{q^2} \tag{5.27}$$

where $\eta(T_i)$ is the standard equilibrium critical exponent and the the relation $2\pi\eta(T_i) = T_i$ valid in the spin-wave approximation was used, see [61, 84, 4].

5.2.2 Non-equilibrium statistical field theory

As before, we introduce a Martin-Siggia-Rose formalism which characterizes the system is term of an action $S[\phi,\widetilde{\phi},\boldsymbol{h}]$ depending on the phase field ϕ and an associated response field $\widetilde{\phi}$ and we also include a (possibly space-dependent) magnetic field $\boldsymbol{h} = \sum_i h_i\boldsymbol{e}_i$. We decompose the action

$$S[\phi,\widetilde{\phi}] = \Sigma[\phi,\widetilde{\phi},\boldsymbol{h}] + \sigma[\phi,\widetilde{\phi}] \tag{5.28}$$

into a bulk term $\Sigma[\phi,\widetilde{\phi},\boldsymbol{h}]$ and an initial term $\sigma[\phi,\widetilde{\phi}]$. These two terms include the thermal and the initial noise. Explicitly

$$\Sigma[\phi,\widetilde{\phi},\boldsymbol{h}] = \int \mathrm{d}t\,\mathrm{d}\boldsymbol{r}\;\widetilde{\phi}\left[\frac{\partial\phi}{\partial t} - \Delta\phi + \sin(\phi)h_1 - \cos(\phi)h_2\right] - T\int\mathrm{d}t\,\mathrm{d}\boldsymbol{r}\;\widetilde{\phi}(t,\boldsymbol{r})\widetilde{\phi}(t,\boldsymbol{r}) \tag{5.29}$$

and

$$\sigma[\phi,\widetilde{\phi}] = -\frac{1}{2}\int\mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{r}'\;\widetilde{\phi}(0,\boldsymbol{r})a(\boldsymbol{r}-\boldsymbol{r}')\widetilde{\phi}(0,\boldsymbol{r}') \tag{5.30}$$

where the function $a(\boldsymbol{r})$ describes the initial conditions according to eq. (5.27).

[7] For quenches from *above* $T_{\rm KT}$ in $2D$, vortex configurations also become important and this leads to logarithmic scaling, see [83] for details.

We now simplify the general expressions for the two-point functions. There is nothing to do for the angular correlation function $C(t,s;\boldsymbol{r},\boldsymbol{r}') = \langle\phi(t,\boldsymbol{r})\phi(s,\boldsymbol{r}')\rangle$ and we start with the magnetic correlation function Γ which is given by

$$\Gamma(t,s;\boldsymbol{r},\boldsymbol{r}') = \int \mathcal{D}\phi\mathcal{D}\widetilde{\phi}\ \cos\left(\phi(t,\boldsymbol{r}) - \phi(s,\boldsymbol{r}')\right) \exp\left(-S[\phi,\widetilde{\phi}]\right) \tag{5.31}$$

For a vanishing magnetic field the bulk action $\Sigma[\phi,\widetilde{\phi},\mathbf{0}]$ is a quadratic form in the fields $\phi,\widetilde{\phi}$ which are therefore Gaussian. Standard techniques explained in appendix A lead to

$$\begin{aligned}\Gamma(t,s;\boldsymbol{r},\boldsymbol{r}') &= \exp\left[-\frac{1}{2}\left\langle(\phi(t,\boldsymbol{r}) - \phi(s,\boldsymbol{r}'))^2\right\rangle\right] &(5.32)\\ &= \exp\left[C(t,s;\boldsymbol{r},\boldsymbol{r}') - \frac{C(t,t;\boldsymbol{r},\boldsymbol{r}) + C(s,s;\boldsymbol{r}',\boldsymbol{r}')}{2}\right] &(5.33)\end{aligned}$$

where in the second line the argument of the exponential was expanded. This gives Γ in terms of angular correlators C.

Next, we consider the response functions. For the angular response R, we quote from the MSR formalism the standard result

$$R(t,s;\boldsymbol{r},\boldsymbol{r}') = \langle\phi(t,\boldsymbol{r})\widetilde{\phi}(s,\boldsymbol{r}')\rangle \tag{5.34}$$

It remains to consider the response of the spin vector $\boldsymbol{S}$ at time t and position $\boldsymbol{r}$ to some magnetic field $\boldsymbol{h}(s,\boldsymbol{r}')$ at time s and position $\boldsymbol{r}'$. From the definition (5.25) we have

$$\rho(t,s;\boldsymbol{r},\boldsymbol{r}') = \lim_{\boldsymbol{h}\to 0}\left[\frac{\langle\langle\cos\phi(t,\boldsymbol{r})\rangle\rangle - \langle\cos\phi(t,\boldsymbol{r})\rangle}{h_1(s,\boldsymbol{r}')} + \frac{\langle\langle\sin\phi(t,\boldsymbol{r})\rangle\rangle - \langle\sin\phi(t,\boldsymbol{r})\rangle}{h_2(s,\boldsymbol{r}')}\right] \tag{5.35}$$

where the average $\langle\langle\cdot\rangle\rangle$ is to be taken with a magnetic field. Expanding the action (5.28) to first order in both components h_1, h_2 of the magnetic field, we find

$$\begin{aligned}\langle\langle\cos\phi(t,\boldsymbol{r})\rangle\rangle &= \langle\cos\phi(t,\boldsymbol{r})\rangle + h_1(s,\boldsymbol{r}')\langle\cos\phi(t,\boldsymbol{r})\sin\phi(s,\boldsymbol{r}')\widetilde{\phi}(s,\boldsymbol{r}')\rangle\\ \langle\langle\sin\phi(t,\boldsymbol{r})\rangle\rangle &= \langle\sin\phi(t,\boldsymbol{r})\rangle - h_2(s,\boldsymbol{r}')\langle\sin\phi(t,\boldsymbol{r})\cos\phi(s,\boldsymbol{r}')\widetilde{\phi}(s,\boldsymbol{r}')\rangle\end{aligned} \tag{5.36}$$

It follows that the response function can be expressed as

$$\rho(t,s;\boldsymbol{r},\boldsymbol{r}') = \langle\widetilde{\phi}(s,\boldsymbol{r}')\sin\left(\phi(t,\boldsymbol{r}) - \phi(s,\boldsymbol{r}')\right)\rangle \tag{5.37}$$

Since both fields $\phi,\widetilde{\phi}$ are Gaussian, it can further be shown that (see appendix A)

$$\rho(t,s;\boldsymbol{r},\boldsymbol{r}') = R(t,s;\boldsymbol{r},\boldsymbol{r}')\Gamma(t,s;\boldsymbol{r},\boldsymbol{r}') \tag{5.38}$$

and we see explicitly that the relationship between R and ρ is non-trivial. That no higher correlators than the magnetic two-point correlation function Γ enter is a consequence of the Gaussian nature of the theory at hand.

Eqs. (5.33,5.34,5.38) are the main results of this subsection. Before we can evaluate them, we need some information on the validity of the spin-wave approximation.

5.2.3 Remarks on the validity of the spin-wave approximation

We need a criterion informing us up to what point the results on Γ and ρ derived in the previous subsection within the spin-wave approximation should be reliable.

The correlation function $C(t,s;\boldsymbol{r},\boldsymbol{r}')$ has already been obtained above and is given in eq. (3.25). For our choice (5.27) of initial conditions, its Fourier transform $\widehat{C}(t,s;\boldsymbol{q})$ is [21, 84]

$$\widehat{C}(t,s;\boldsymbol{q}) = (T_i - T_f)\widehat{G}(t+s;\boldsymbol{q}) + T_f\widehat{G}(t-s;\boldsymbol{q}) \tag{5.39}$$

where $\widehat{G}$ is given by

$$\widehat{G}(u;\boldsymbol{q}) := \frac{1}{\boldsymbol{q}^2}\exp\left(-\boldsymbol{q}^2(u+\Lambda^2)\right) \tag{5.40}$$

and we have explicitly introduced an UV-cutoff which simulates the lattice spacing (we shall let $\Lambda \to 0$ at the end). Therefore, a two-point correlation function $\langle\phi\phi\rangle$ is of order $\mathrm{O}(T_i, T_f)$ whereas a response function $\langle\phi\widetilde{\phi}\rangle$ is of order $\mathrm{O}(1)$ in the initial and final temperatures.

In order to discuss further the validity of the spin-wave approximation, we keep the next-order term as well and consider the Hamiltonian

$$\mathcal{H}[\phi] = \mathcal{H}_0 + \frac{1}{2}\int \mathrm{d}\boldsymbol{r}\ \left[(\nabla\phi(\boldsymbol{r}))^2 + g_4\,(\nabla\phi(\boldsymbol{r}))^4\right] \tag{5.41}$$

where g_4 is some constant. A straightforward calculation shows that to first order in g_4, the correction to the spin-wave approximation of the two-point correlation function is given by

$$\delta C(t,s;\boldsymbol{r},\boldsymbol{r}') = \int \mathrm{d}u\,\mathrm{d}\boldsymbol{R}\ \left\langle \phi(t,\boldsymbol{r})\phi(s,\boldsymbol{r}')\widetilde{\phi}(u,\boldsymbol{R})\nabla\left(\nabla\phi(u,\boldsymbol{R})\right)^3\right\rangle \simeq \mathrm{O}\left(\left(\frac{T_i}{T_c(d)}\right)^2, \left(\frac{T_f}{T_c(d)}\right)^2\right) \tag{5.42}$$

where the six-point function is factorized into two-point function by Wick's theorem. As a result, *the spin-wave approximation is a first-order approximation in the initial and final temperatures.* Consistent expressions of two-point functions must be expanded to first order in T_i, T_f. Higher-order terms in $T_{i,f}$ calculated within the spin-wave approximation should not be expected to be reliable.

5.2.4 Correlation and response functions in the spin-wave approximation

Finally we are ready to list the result for two-time correlation and response functions in the spin-wave approximation. From the previous subsection we know that C must be of first order in T_i, T_f. The consistent result for the magnetic correlation function is therefore obtained by expanding eq. (5.33) to first order in temperature. Then

$$\Gamma(t,s;\boldsymbol{r},\boldsymbol{r}') \simeq 1 + C(t,s;\boldsymbol{r},\boldsymbol{r}') - \frac{1}{2}\left[C(t,t;\boldsymbol{r},\boldsymbol{r}) + C(s,s;\boldsymbol{r}',\boldsymbol{r}')\right] \tag{5.43}$$

A more suggestive form of this is found as follows. We have

$$\begin{aligned}\langle\boldsymbol{S}(t,\boldsymbol{r})\rangle\cdot\langle\boldsymbol{S}(s,\boldsymbol{r}')\rangle &= \langle\cos\phi(t,\boldsymbol{r})\rangle\langle\cos\phi(s,\boldsymbol{r}')\rangle + \langle\sin\phi(t,\boldsymbol{r})\rangle\langle\sin\phi(s,\boldsymbol{r}')\rangle \\ &\simeq 1 - \frac{1}{2}\langle\phi(t,\boldsymbol{r})^2\rangle\langle\phi(s,\boldsymbol{r}')^2\rangle + \langle\phi(t,\boldsymbol{r})\rangle\langle\phi(s,\boldsymbol{r}')\rangle + \cdots \end{aligned} \tag{5.44}$$

where in the second line we performed a low-temperature expansion which must be kept to second order in ϕ in order to be of first order in the temperature, since $C = \langle\phi\phi\rangle = \mathrm{O}(T)$. Furthermore, because

of the $\phi \mapsto -\phi$ inversion symmetry, $\langle\phi\rangle = 0$. Inserting this into (5.43) we find, of course only in the context of the spin-wave approximation

$$\langle \boldsymbol{S}(t,\boldsymbol{r}) \cdot \boldsymbol{S}(s,\boldsymbol{r}') \rangle - \langle \boldsymbol{S}(t,\boldsymbol{r}) \rangle \cdot \langle \boldsymbol{S}(s,\boldsymbol{r}') \rangle = C(t,s;\boldsymbol{r},\boldsymbol{r}') \tag{5.45}$$

and the relation between Γ and C is finally clarified (the equilibrium version of this is well-known, see [55], sect. 4.2.2).

For notational simplicity, we shall now concentrate on the autocorrelation and autoresponse functions. First, the angular correlation function is given by eqs. (5.39) and (5.40). We have

$$C(t,s) = \begin{cases} \frac{2(4\pi)^{-d/2}}{d-2}\left[(T_i - T_f)(t+s+\Lambda^2)^{1-d/2} + T_f(t-s+\Lambda^2)^{1-d/2}\right] & ;\ \text{if } d > 2 \\ (4\pi)^{-1}\left[(T_f - T_i)\ln(t+s+\Lambda^2) - T_f \ln(t-s+\Lambda^2)\right] & ;\ \text{if } d = 2. \end{cases} \tag{5.46}$$

Second, the autoresponse functions are given by

$$\begin{aligned} \rho(t,s) &= R(t,s)\left(1 + C(t,s) - \frac{1}{2}\left[C(t,t) + C(s,s)\right]\right) \\ R(t,s) &= \left[4\pi(t-s+\Lambda^2)\right]^{-d/2} \end{aligned} \tag{5.47}$$

These results require a detailed discussion.

1. The $2D$ XY model was studied in detail in the spin-wave approximation before [84, 4] and we now show that their results, although they might at first sight appear to be different, agree with eqs. (5.43,5.46,5.47). For notational simplicity, we restrict to $T_i = 0$, $T_f = T$. First, the magnetic autocorrelation function is [4, eq. (11)]

$$\begin{aligned} \Gamma(t,s) = \langle \boldsymbol{S}(t,\boldsymbol{r}) \cdot \boldsymbol{S}(s,\boldsymbol{r}) \rangle &= \left(\frac{\Lambda^4(t+s+\Lambda^2)^2}{(2t+\Lambda^2)(2s+\Lambda^2)(t-s+\Lambda^2)^2}\right)^{\eta(T)/4} \\ &= \exp\left[\frac{\eta(T)}{4}\ln\left(\frac{\Lambda^4(t+s+\Lambda^2)^2}{(2t+\Lambda^2)(2s+\Lambda^2)(t-s+\Lambda^2)^2}\right)\right] \\ &\simeq 1 + \frac{T}{4\pi}\left(\ln(t+s+\Lambda^2) - \ln(t-s+\Lambda^2) - \frac{1}{2}\ln(2t+\Lambda^2) - \frac{1}{2}\ln(2s+\Lambda^2) + \ln\Lambda^2\right) + \mathrm{O}(T^2) \end{aligned} \tag{5.48}$$

since the spin-wave approximation is only consistent to lowest order in T. It is now clear that the above result is reproduced by inserting $C(t,s)$ from eq. (5.46) with $d = 2$ into (5.43). Second, the linear spin response in $2D$ is [4, eq. (13)]

$$\begin{aligned} \rho(t,s) &= \lim_{\boldsymbol{h}\to 0}\frac{\delta\langle \boldsymbol{S}(t,\boldsymbol{r})\rangle}{\delta \boldsymbol{h}(s,\boldsymbol{r})} = \frac{1}{4\pi(t-s+\Lambda^2)}\left(\frac{\Lambda^4(t+s+\Lambda^2)^2}{(2t+\Lambda^2)(2s+\Lambda^2)(t-s+\Lambda^2)^2}\right)^{\eta(T)/4} \\ &= \frac{1}{4\pi(t-s+\Lambda^2)}\,\Gamma(t,s) \end{aligned} \tag{5.49}$$

in agreement with eq. (5.47) with $d = 2$, as it should be.

In $2D$, the results for $\Gamma(t,s)$ and $\rho(t,s)$ were confirmed by a simulational study with an ordered initial state and $T_f < T_c$ [1].

2. For $T_i = T_f$, the two-point functions are stationary. This is only to be expected, since in this case the system was prepared in an equilibrium state and remains there.

3. Ageing occurs when $T_i \neq T_f$. Time-translation invariance is broken and we proceed to analyse the resulting scaling behaviour. Now, there are in principle two equally appealing sets of variables. First, we may choose to work with the angular correlation function C and its associated response R. Recalling the scaling forms introduced in section 1 (see eqs. (1.2,1.3)) we shall characterize them by the exponents $a, b, \lambda_C, \lambda_R$. Second, we may prefer instead to work with the magnetic correlation function Γ and its associated response ρ. We shall use the same scaling forms, but for clarity we shall denote the corresponding exponents by $a', b', \lambda'_C, \lambda'_R$. These exponents are straightforwardly read off in the ageing regime where $t, s, t-s \gg \Lambda$ and we collect the results in table 1. The exponent $\lambda'_C = (\eta_i + \eta_f)/2$ was already known [84].

Table 1: Ageing exponents of the d-dimensional XY model in the spin-wave approximation. Here $\eta_{i,f} = \eta(T_{i,f}) = T_{i,f}/(2\pi)$ describe the initial and final correlation exponents.

	angular correlation and response			
	a	b	λ_C	λ_R
$d=2$	0	0	0	2
$d>2$	$d/2-1$	$d/2-1$	d	d
	magnetic correlation and response			
	a'	b'	λ'_C	λ'_R
$d=2$	$\eta_f/2$	$\eta_f/2$	$(\eta_i+\eta_f)/2$	$2+(\eta_i+\eta_f)/2$
$d>2$	$d/2-1$	$d/2-1$	d	d

We see that the exponents satisfy the equalities $a = b$ and $a' = b'$ expected for nonequilibrium critical dynamics and point out that in $2D$, the autocorrelation and autoresponse exponents are different: $\lambda_R - \lambda_C = \lambda'_R - \lambda'_C = 2$. This effect comes from the non-disordered initial condition of the spins, as explained first in [77].

4. Having discussed the values of the ageing exponents, we wish to compare the form of the scaling functions with the predictions of local scale-invariance as derived in section 4. This requires, however, the correct identification of the quasiprimary fields in our system, see section 2. It is *only* the quasiprimary fields which are expected to transform in a simple way under a local scale-transformation and the transformation laws of more complicated fields built from quasiprimary fields must derived accordingly.

   In the case at hand, it is clear from the complicated structure of the magnetic correlation and response functions that the magnetic order parameter $\boldsymbol{S}(t, \boldsymbol{r})$ does *not* correspond to a quasiprimary field. Rather, the quasiprimary field should be identified with the phase $\phi(t, \boldsymbol{r})$. Indeed, the form of the angular response $R(t, s)$ is in perfect agreement with the prediction (4.8,4.17) of local scale-invariance. This suggests that the response field $\widetilde{\phi}(s, \boldsymbol{r})$ should be quasiprimary as well.

5. Having thus identified ϕ and $\widetilde{\phi}$ as quasiprimary fields of the model, it is now clear that the angular autocorrelation function $C(t, s)$ should be compared to the critical dynamics correlation function as derived in section 4. However, a direct comparison with eq. (4.34) is not possible, since in its derivation fully disordered initial conditions were assumed.

   We shall therefore proceed in two steps. First, we shall consider the case $T_i = 0$. Because of our initial conditions (5.27) the initial correlator then vanishes and eq. (4.34) should now hold true. Second, we shall show that in the context of the free-field theory underlying the spin-wave approximation of the XY model, the restriction to uncorrelated initial states can be lifted.

We now set $T_i = 0$ and $T_f = T$. From the exponents in table 1, the predicted autocorrelator scaling function follows from (4.34) as

$$f_C(y) = \Phi_0 \int_0^1 \mathrm{d}\theta \ (y + 1 - 2\theta)^{-d/2} \quad ; \quad d \geq 2 \tag{5.50}$$

and we see immediately that this is in agreement with the explicit angular correlator (5.46), upon identification of Φ_0.

Finally, if we also allow for $T_i > 0$, there is a contribution to $C(t,s)$ from the initial condition. We then return to the basic result (4.9) and decompose $C(t,s) = C_{th} + C_{pr}$. The thermal term C_{th} was treated before and the preparation term is analysed in appendix B, with the result

$$C_{pr} = \frac{2(4\pi)^{-d/2}}{d-2} T_i (t+s)^{1-d/2} \quad ; \quad d > 2 \tag{5.51}$$

in complete agreement with (5.46). The case $d = 2$ is treated similarly.

In conclusion, the two-time autocorrelation and autoresponse functions of the XY model treated in spin-wave approximation are in perfect agreement with local scale-invariance, *provided* the angular variable ϕ and its associated response field are identified as the quasiprimary fields of the model.

5.3 Fluctuation-dissipation relations in the XY model

Having checked that both correlation and response functions agree with the local scale-invariance prediction, we now inquire what can be said on the approach of the model towards equilibrium. A convenient way to study this is through the so-called fluctuation-dissipation ratio [21], see section 1. Since we have seen that in the XY model angular and magnetic observables behave quite differently, it is convenient to define two distinct fluctuation-dissipation ratios, namely

$$\begin{aligned} \Xi(t,s) &:= T_f \rho(t,s) \left(\frac{\partial \Gamma(t,s)}{\partial s} \right)^{-1} \\ X(t,s) &:= T_f R(t,s) \left(\frac{\partial C(t,s)}{\partial s} \right)^{-1} \end{aligned} \tag{5.52}$$

Of particular interest will be the limit fluctuation-dissipation ratios X_∞ defined in eq. (1.5) and similarly Ξ_∞. We have seen before that the scaling of autocorrelation and autoresponse functions is according to the expectations of nonequilibrium critical dynamics. In this case, according to the Godrèche-Luck conjecture [38], X_∞ and Ξ_∞ should be universal.

We shall use the available exact results in the XY model to test this conjecture by studying the dependence of X and Ξ on the ratio $\alpha := T_i/T_f$ of initial and final temperatures.

5.3.1 Fluctuation-dissipation ratio for magnetic variables

The fluctuation-dissipation ratio $\Xi(t,s)$ obtained from the magnetic correlation and response functions reads, with $y = t/s$

$$\frac{1}{\Xi(y)} = 1 + \left(1 - \frac{T_i}{T_f}\right) \left[\left(\frac{y-1}{y+1}\right)^{d/2} - \left(\frac{y-1}{2}\right)^{d/2} \right] \tag{5.53}$$

For $T_i = 0$ and $d = 2$ this was already known [4]. In the quasiequilibrium regime $y \simeq 1$ the fluctuation-dissipation theorem should hold. Indeed we find $\lim_{y\to 1} \Xi(y) = 1$ which confirms that Ξ should be well-defined. For large values of y, that is for well-separated times, we have

$$\Xi(y) \simeq \frac{T_f}{T_i - T_f} \left(\frac{y-1}{2}\right)^{-d/2} \tag{5.54}$$

and therefore $\Xi_\infty = 0$, indeed a universal constant. It is remarkable that the asymptotic value of $\Xi(y)$ should be independent of d and that is agrees with the value $\Xi_\infty = 0$ of phase-ordering kinetics in the low-temperature phase with an ordered, non-critical equilibrium state. This kind of result should be more typical of an ordered ferromagnetic equilibrium state as it occurs for $d > 2$ but is not really expected for $d = 2$ since the equilibrium $2D$ XY model is critical even below $T_{\rm KT}$.

Finally, for large y the asymptotic form of $\Xi(y)$ is independent on whether the system is cooled or heated. The temperatures merely enter into a scaling amplitude.

5.3.2 Fluctuation-dissipation ratio for angular variables

In the same way, the fluctuation-dissipation ratio for the angular variables is found. It reads

$$\frac{1}{X(y)} = 1 + \left(1 - \frac{T_i}{T_f}\right)\left(\frac{y+1}{y-1}\right)^{-d/2} \tag{5.55}$$

As before, in the quasiequilibrium regime $t \simeq s$, $X(t,s) = 1$. Surprisingly, however, for large values of $y = t/s$, the limit fluctuation-dissipation ratio

$$X_\infty = \left(2 - \frac{T_i}{T_f}\right)^{-1} \tag{5.56}$$

depends continuously on $\alpha = T_i/T_f$. We recall from table 1 that the non-equilibrium exponents of the *angular* variables are all independent of both T_i and T_f and although the exponents do depend on d, we see from (5.56) that X_∞ does not. Taken literally, this would be an example of a non-universal value of the limit fluctuation-dissipation ratio.

We recall that most 'physically reasonable' systems undergoing nonequilibrium critical dynamics one usually finds $0 \leq X_\infty \leq 1/2$, see [39] for a review. Motivated from mean-field theories of spin glasses, it is sometimes suggested that $T_{\rm eff} := T/X_\infty$ might be interpreted as an effective temperature for which the fluctuation-dissipation theorem would hold. It is hard to see how in this case (X_∞ may even become negative) such an interpretation could be maintained.

6 The critical voter model in d dimensions

We now study a qualitatively different type of application of local scale-invariance in the so-called voter model, see [64] and references therein. The model describes the temporal evolution of configurations $\mathcal{C}$ of spins $\sigma_{\boldsymbol{r}}(t) = \pm 1$ on a d-dimensional hypercubic lattice $\mathbb{Z}^d$. The dynamics is assumed to be given by a master equation

$$\frac{\mathrm{d}}{\mathrm{d}t} P(\mathcal{C};t) = \sum_{\boldsymbol{r}\in\mathbb{Z}^d} \left[W_{\boldsymbol{r}}(\mathcal{C}^{(\boldsymbol{r})}) P(\mathcal{C}^{(\boldsymbol{r})};t) - W_{\boldsymbol{r}}(\mathcal{C}) P(\mathcal{C};t) \right] \tag{6.1}$$

Here the configuration $\mathcal{C}^{(\boldsymbol{r})}$ is obtained from $\mathcal{C}$ by inverting the single spin at site $\boldsymbol{r}$. Finally, the transition rates for a spin reversal $\sigma_{\boldsymbol{r}} \mapsto -\sigma_{\boldsymbol{r}}$ are given by

$$W_{\boldsymbol{r}}(\mathcal{C}) = \frac{1}{2}\left[1 - \frac{1}{2d}\sum_{k=1}^{d}\left(\sigma_{\boldsymbol{r}}\sigma_{\boldsymbol{r}+\boldsymbol{e}_k} + \sigma_{\boldsymbol{r}}\sigma_{\boldsymbol{r}-\boldsymbol{e}_k}\right)\right] \tag{6.2}$$

where the $\boldsymbol{e}_k$, $k = 1, \dots d$ form an orthonormalized basis of unit vectors on the d-dimensional hypercubic lattice.

With respect to the kinetic spherical model and the XY model studied previously, the voter model is different since in general it does *not* satisfy detailed balance and therefore will not relax to an equilibrium state. By considering a general kinetic Ising model with a dynamics respecting the global $\mathbb{Z}_2$-symmetry, it can be shown that the transition rates (6.2) correspond to the critical point of the so-called linear voter model [73, 88]. The non-equilibrium kinetics of the critical voter model (6.2) has been studied in detail by Dornic [26]. In $d = 1$ dimensions the model coïncides with the kinetic Glauber-Ising model at zero temperature (which we shall revisit in appendix C) but for $d > 1$ these two models are different. In particular, it is known that the domain growth of the voter model is not driven by the minimization of the surface tension between the two phases [25, 26] but which is the mechanism which drives ageing in simple ferromagnets [9, 11].

We are interested here in the correlation functions $C_{\boldsymbol{r}}(t) = \langle\sigma_{\boldsymbol{r}}(t)\sigma_{\boldsymbol{0}}(t)\rangle$ and $C_{\boldsymbol{r}}(t,s) = \langle\sigma_{\boldsymbol{r}}(t)\sigma_{\boldsymbol{0}}(s)\rangle$ which are easily seen to satisfy the following equations of motion [28, 26]

$$\begin{aligned} \frac{\partial}{\partial t}C_{\boldsymbol{r}}(t) &= 2\Delta_{\boldsymbol{r}}C_{\boldsymbol{r}}(t) \\ \frac{\partial}{\partial t}C_{\boldsymbol{r}}(t,s) &= \Delta_{\boldsymbol{r}}C_{\boldsymbol{r}}(t,s) \end{aligned} \tag{6.3}$$

subject to the following boundary conditions

$$C_{\boldsymbol{0}}(t) = 1 \;\;,\;\; C_{\boldsymbol{r}}(0) = \delta_{\boldsymbol{r},\boldsymbol{0}} \;\;,\;\; C_{\boldsymbol{r}}(t,t) = C_{\boldsymbol{r}}(t) \tag{6.4}$$

where $\Delta_{\boldsymbol{r}}$ is the discrete Laplacian and where the initial magnetization $\sum_{\boldsymbol{r}}\langle\sigma_{\boldsymbol{r}}(0)\rangle = 0$. For the autocorrelation function $C(t,s) = C_{\boldsymbol{0}}(t,s)$ of the critical voter model (6.2) one finds in the ageing regime, where t, s and $t-s$ are all sufficiently large [26], with $y = t/s$

$$C(t,s) = \begin{cases} \frac{2}{\pi}\arctan\sqrt{2/(y-1)} & ;\ \text{if } d = 1 \\ \ln(s)^{-1}\,\ln\left((y+1)/(y-1)\right) & ;\ \text{if } d = 2 \\ s^{-d/2+1}\left(\frac{d}{2\pi}\right)^{d/2}\frac{2\gamma_d}{d-2}\left[(y-1)^{-d/2+1} - (y+1)^{-d/2+1}\right] & ;\ \text{if } 2 < d < 4 \end{cases} \tag{6.5}$$

where γ_d is the probability that a random walk in d dimensions and starting from the origin never returns. We are not aware of published results on $R(t,s)$ for $2 < d < 4$ in this model. Clearly, time-translation invariance is broken for all $d \geq 2$.

We wish to compare (6.5) with the prediction eq. (4.34) of local scale-invariance. The case $d = 1$ will be dealt with in appendix C and since for $d = 2$ logarithmic scaling is found, the form of local scale-invariance as presented here is not applicable.[8] We therefore concentrate on the dimensions $2 < d < 4$.

[8] An extension of Schrödinger invariance to logarithmic Schrödinger invariance in analogy to logarithmic conformal invariance, see e.g. [44], might be needed here.

From the asymptotics of $C(t,s)$ in (6.5) we read off the exponents

$$b = \frac{d}{2} - 1 \quad , \quad \lambda_C = d \tag{6.6}$$

since the exponent $z = 2$ is known, see [64, 26]. From eq. (4.34) we expect

$$f_C(y) = \Phi_{0,c} \int_0^1 \mathrm{d}\theta \; (y + 1 - 2\theta)^{-d/2} = \frac{\Phi_{0,c}}{d-2} \left[(y-1)^{-d/2+1} - (y+1)^{-d/2+1} \right] \tag{6.7}$$

in full agreement with (6.5) and we identify the normalization constant $\Phi_{0,c} = 2\gamma_d \left(d/2\pi\right)^{d/2}$. Indeed, we see that the form (4.33) – which in principle is only valid asymptotically – is in fact exact in the critical voter model. This is not surprising in view of the underlying free-field theory.

In conclusion, for the critical voter model with two competing steady states ageing occurs. The scaling form of the two-time autocorrelation function is in exact agreement with the prediction of local scale-invariance. This is the first time such an agreement is found for a system without detailed balance.

7 The free random walk

Last, but not least, we briefly consider the simplest example of a system undergoing ageing: the free random walk [21]. The Langevin equation describing the time-evolution of the order parameter ϕ reads

$$\frac{\partial \phi(t)}{\partial t} = h(t) + \eta(t) \tag{7.1}$$

where a deterministic external field $h(t)$ has been added in order to be able to compute response functions. The Gaussian noise η is characterized as usual by its first two moments, see eq. (1.9). Here, we choose the notation such that the relationship to local scale-invariance becomes evident.

The autocorrelation and linear autoresponse functions were already calculated by Cugliandolo et *al.* [21]. They obtained, with the initial condition $C(t,0) = 0$

$$\begin{aligned} R(t,s) &= \left.\frac{\delta\langle\phi(t)\rangle}{\delta h(s)}\right|_{h=0} = \frac{1}{2T}\langle\phi(t)\eta(s)\rangle = \Theta(t-s) \\ C(t,s) &= \langle\phi(t)\phi(s)\rangle = 2T\min(t,s) \end{aligned} \tag{7.2}$$

Clearly, the system undergoes ageing, since $C(t,s)$ does not merely depend on $t-s$ and furthermore, it stays forever out of equilibrium, since the fluctuation-dissipation ratio $X(t,s) = 1/2$ [21].

We wish to check that the results (7.2) are compatible with the predictions of local scale invariance. For the autoresponse function, the first eq. (7.2) is clearly compatible with (4.7,4.17,4.18), with the exponents $a = -1$ and $\lambda_R = 0$. For the autocorrelation function, we expect from (4.12)

$$C(t,s) = T \int \mathrm{d}u \; \left.\frac{\delta^2\langle\phi(t)\phi(s,)\rangle}{\delta h(u)^2}\right|_{h=0} \tag{7.3}$$

where in view of the initial condition $C(t,0) = 0$ used in eq. (7.2) we assumed a vanishing initial correlator.

In order to calculate the above derivative, we solve the Langevin equation for a given field h and obtain the autocorrelation function

$$C(t,s;[h]) = C(t,s;[0]) + \int_0^s \int_0^t \mathrm{d}v\mathrm{d}w \; h(v)h(w) \tag{7.4}$$

where $C(t,s;[0]) = C(t,s)$ is of course given by the second eq. (7.2). The required second derivative of the autocorrelation function becomes in a field-theoretic formulation some three-point correlator (see eq. (4.10)) and it is now easy to see that

$$R_0^{(3)}(t,s,u) = \left.\frac{\delta^2 C(t,s;[h])}{\delta h(u)^2}\right|_{h=0} = 2\Theta(t-u)\Theta(s-u) \tag{7.5}$$

Inserting this into (7.3), the desired result for $C(t,s)$ in eq. (7.2) is indeed recovered. We read off the exponents $b=-1$ and $\lambda_C = 0$. Of course, the exponent equalities $a=b$ and $\lambda_C = \lambda_R$ are a necessary requirement for having a non-vanishing limit fluctuation-dissipation ratio $X_\infty = 1/2$.

In conclusion, the evidence from the two-time autocorrelation and autoresponse function of the free random walk is fully consistent with local scale-invariance.

8 Conclusions and discussion

We have analysed the ageing behaviour in systems with a non-conserved order parameter and described by a Langevin equation. Our main assumption was that the noiseless part of that Langevin equation is Galilei-invariant. Together with dynamical scaling this hypothesis fixes the dynamical exponent $z=2$ and implies for local theories Schrödinger-invariance [48]. There are good reasons for admitting such a hypothesis. For example the phase-ordering kinetics of the Glauber-Ising model in $d>1$ dimensions quenched to a temperature $T<T_c$ provides strong evidence that the scaling function of its space-time response function $R(t,s;\boldsymbol{r},\boldsymbol{r}')$ has the form predicted from Galilei-invariance [49]. However, since groups of local scale transformations such as the Schrödinger group are dynamical symmetries of noiseless differential equations only, the rôle of the noise in the Langevin equation or from the initial conditions has to be addressed.

We have carried out such a study, for the important special case where $z=2$ and the initial state is fully disordered. Considering the Langevin equation as the classical equation of motion of a MSR-type field theory, we have calculated the two-time correlation and linear response functions by studying how the dynamical symmetry properties of the noiseless part of that field theory are reflected in these noisy averages. These averages can be written in form of a perturbative expansion around the noiseless theory and we have shown that only a *finite* number of terms in these series contributes. Specifically, we have found:

1. The two-time linear response function $R = \langle\phi\widetilde{\phi}\rangle$ involving only quasiprimary fields is independent of both the thermal and the initial noise. This explains why the form (1.7) of scaling function $f_R(y)$ of the linear response – previously derived from the symmetries of the noiseless theory – has been reproduced in many different systems with $T>0$ either exactly or with a considerable numerical precision [12, 16, 21, 38, 39, 45, 46, 47, 49, 58, 69, 77].

2. We obtain the reduction formula eq. (4.9) which expresses the two-time correlation function in terms of certain noiseless three- and four-point response functions. For the uncorrelated initial conditions (4.11) only a single noiseless three-point response function is needed.

3. The scaling forms of correlation and response functions are governed by the two non-trivial exponents λ_C and λ_R which are in general distinct from each other. Given the initial correlator (4.11), local scale-invariance with $z=2$ provides a sufficient condition for the exponent equality

$$\lambda_C = \lambda_R \tag{8.1}$$

4. The scaling of the two-time autocorrelator $C(t,s)$ is described by a scaling function $\Phi(w)$ which in turn depends on a scaling function $\Psi(\rho)$ which arises in the three-point function of quasiprimary fields in Schrödinger-invariant theories. Depending on whether the thermal or the initial noise is dominant, two distinct scaling forms eqs. (4.25) and (4.32) are found and we have argued that they should describe the cases when the system is quenched to temperatures $T < T_c$ and $T = T_c$, respectively (in the latter case only if $z = 2$ still holds after renormalization).

5. Schrödinger-invariance by itself does not determine the form of $\Psi(\rho)$. We have argued that the related three-point response function should be non-singular and then find the asymptotic behaviour for $w \to \infty$

$$\Phi(w) \sim w^{-\varphi} \;\;,\;\; \varphi = \left\{ \begin{array}{ll} d/2 - \lambda_C & ;\; \text{if } T < T_c \\ 1 + a & ;\; \text{if } T = T_c \end{array} \right. \tag{8.2}$$

   This suggests the following approximate forms, with $y = t/s$

$$C(t,s) \approx \left\{ \begin{array}{ll} M_{\rm eq}^2 \left[(y+1)^2/(4y)\right]^{-\lambda_C/2} & ;\; \text{if } T < T_c \\ \\ \Phi_{0,c}\, y^{1+a-\lambda_C/2} \int_0^1 \mathrm{d}\theta\, \theta^{\lambda_C-2-2a}(y+1-2\theta)^{-1-a} & ;\; \text{if } T = T_c \end{array} \right. \tag{8.3}$$

   At least, these forms are consistent with the required asymptotic behaviour of $f_C(y)$ as $y \to \infty$, see section 1. For free-field theories (8.2) holds for all values of w and then (8.3) becomes exact.

   In the past, approximate expressions for the scaling of magnetic correlation functions were derived from Gaussian closure procedures for kinetic O(n)-models undergoing phase-ordering kinetics, see [9]. This gives for the magnetic autocorrelation function at $T = 0$ [7, 8, 83]

$$f_{C,\rm BPT}(y) \approx \frac{n}{2\pi} \left[B\left(\frac{1}{2}, \frac{n+1}{2}\right)\right]^2 \left(\frac{4y}{(y+1)^2}\right)^{d/4} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{n+2}{2}; \left(\frac{4y}{(y+1)^2}\right)^{d/2}\right) \tag{8.4}$$

   where B is Euler's beta function and ${}_2F_1$ a hypergeometric function. It is easy to see that the scaling function (8.3) with $T = 0$ is recovered from (8.4) in the $n \to \infty$ limit. In the notation of section 5.2, eq. (8.4) implies an exponent $\lambda_C' = d/2$. This is a typical value for a free-field theory for $T < T_c$ (which indeed describes the O(n)-model in the $n \to \infty$ limit) but which will in general not hold for $n < \infty$ and thus (8.4) cannot be expected to represent well the behaviour for $y = t/s$ large.[9] One might wonder whether the long-standing difficulties in arriving at scaling functions which cover adequately the whole range of values of y should not be related at least partially to having worked with dynamic variables which might turn out to be not the most basic ones of the model ?

6. We have tested these predictions on the exact solutions of the kinetic spherical model and the XY model in the spin-wave approximation. In these cases, the exponent a is given by (1.4). In order to compare the exact model results with the prediction (8.3), it was necessary to carefully identify the quasiprimary fields of the models. For the spherical model, the natural magnetic order parameter and its response field could be used as quasiprimary fields. On the other hand, for the XY model it turned out that the coarse-grained magnetic moment is not quasiprimary but rather the angular variable is.

[9] Indeed, to leading order in $1/n$ and for uncorrelated initial conditions, $\lambda_C' = d/2 + (4/3)^d (d+2)\frac{2d}{9} B(1+d/2, 1+d/2) n^{-1}$ [6].

These examples underline the importance of the correct identification of the quasiprimary fields in a given model.

The explicit results in the XY model also illuminate in a new way the Godrèche-Luck conjecture on the universality of the limit fluctuation-dissipation ratio. Further tests on the form of the autocorrelation function in the Glauber-Ising model in $d \geq 2$ dimensions are presently carried out and will be reported elsewhere [79, 96].

7. We also showed that the autocorrelator in the critical voter model in $2 < d < 4$ dimensions and which does not satisfy detailed balance, again agrees with (8.3). This is the first example of a new domain of application of local scale-invariance. We also confirmed (8.3) for the free random walk.

   These four confirmations, although all based on an underlying free-field theory provide further evidence in favour of a Galilei-invariance of the noiseless theory.

8. The scaling of the linear response of the $1D$ Glauber-Ising model at $T = 0$ can only be explained through a generalization of the representations of the Lie algebra of local scale-invariance. It would be interesting to see whether a similar phenomenon could be found in different $1D$ systems undergoing ageing at $T = 0$.

Our approach has been based in an essential way on the assumption of Galilei-invariance of the noiseless theory. However, Mazenko [67] recently studied phase-ordering kinetics in the context of the time-dependent non-linear Ginzburg-Landau equation. He carried out a second-order perturbative calculation around a Gaussian theory which is equivalent to the Ohta-Jasnow-Kawasaki approximation and reports a deviation of the two-time autoresponse function $R(t, s)$ from the local scale-invariance prediction (1.7). A similar difficulty had been observed before by Calabrese and Gambassi who studied non-equilibrium dynamics (that is $T = T_c$) of the $\mathrm{O}(n)$ model, for both model A and model C dynamics, through MSR field theory [13, 14, 15]. Again, already their classical action is manifestly not Galilei-invariant.[10]

At face value, Mazenko's result [67] is in disagreement with the simulational data obtained from the $2D$ and $3D$ Glauber-Ising model with $T < T_c$ and based on the master equation. These do reproduce (1.7) for the autoresponse function $R(t, s)$ [47, 49, 52] as well as the extension for the spatio-temporal response $R(t, s; \boldsymbol{r} - \boldsymbol{r}')$ [49]. Could this mean that there are subtle differences between the formulation of stochastic systems either through a master equation or else through a coarse-grained Langevin equation[11] and which affect the formal Galilei-invariance of the theory ? Alternatively, if under renormalization the dynamical exponent $z = 2$ remains constant, might the theory flow to a fixed point where asymptotically Galilei-invariance would hold ?

All in all, based on a postulated extension of dynamical scaling to some local scale-invariance, we have reformulated the problem of finding the scaling function of the two-time autocorrelation function of ageing systems as one of a discussion of the properties of certain three-point response functions of the noiseless theory. The evidence available at present suggests that this approach might be capable of shedding a new light on the issue. Finally, it might be of interest to search for extensions for dynamical exponents $z \neq 2$ and/or to study ageing systems with a conserved order parameter. Asymptotic information on the two-point functions in the latter case is now becoming available [40].

[10] At criticality, the combined effect of thermal and initial fluctuations leads for interacting theories to a non-trivial value of $z \neq 2$ under renormalization so that our arguments are no longer directly applicable.

[11] In $1D$ and at $T = 0$, the autocorrelation exponent $\lambda_C = 1$ found in the Glauber-Ising model [37, 65] differs from the exactly known exponent $\lambda_C = 0.6006\ldots$ [10] determined in the time-dependent Landau-Ginzburg equation.

Appendix A. On Gaussian integration

We present the details of the calculations of the magnetic two-time correlation function Γ and its associated linear response function ρ for the d-dimensional XY model in the spin-wave approximation. They are defined as

$$\begin{aligned}\Gamma(t,s;\boldsymbol{r},\boldsymbol{r}') &= \int \mathcal{D}\phi\mathcal{D}\widetilde{\phi}\;\cos(\phi(t,\boldsymbol{r})-\phi(s,\boldsymbol{r}'))\exp\left(-S[\phi,\widetilde{\phi}]\right)\\ \rho(t,s;\boldsymbol{r},\boldsymbol{r}') &= \int \mathcal{D}\phi\mathcal{D}\widetilde{\phi}\,\widetilde{\phi}(s,\boldsymbol{r}')\sin\left(\phi(t,\boldsymbol{r})-\phi(s,\boldsymbol{r}')\right)\exp\left(-S[\phi,\widetilde{\phi}]\right)\end{aligned} \tag{A1}$$

where $S[\phi,\widetilde{\phi}]$ is the free-field Martin-Siggia-Rose action which is Gaussian.

A.1 The correlation function $\Gamma(t,s;\boldsymbol{r},\boldsymbol{r}')$

Using de Moivre's identities

$$\begin{aligned}\Gamma(t,s;\boldsymbol{r},\boldsymbol{r}') &= \frac{1}{2}\left[\langle\exp\mathrm{i}(\phi(t,\boldsymbol{r})-\phi(s,\boldsymbol{r}'))\rangle+\langle\exp\mathrm{i}(\phi(s,\boldsymbol{r}')-\phi(t,\boldsymbol{r}))\rangle\right]\\ &= \frac{1}{2}\left[\left\langle\exp\mathrm{i}\int\mathrm{d}u\,\mathrm{d}\boldsymbol{R}\,J(u,\boldsymbol{R})\phi(u,\boldsymbol{R})\right\rangle+\left\langle\exp-\mathrm{i}\int\mathrm{d}u\,\mathrm{d}\boldsymbol{R}\,J(u,\boldsymbol{R})\phi(u,\boldsymbol{R})\right\rangle\right]\end{aligned} \tag{A2}$$

where $J(u,\boldsymbol{R})$ is given by

$$J(u,\boldsymbol{R}) = \delta(u-t)\delta(\boldsymbol{R}-\boldsymbol{R}) - \delta(u-s)\delta(\boldsymbol{R}-\boldsymbol{r}') \tag{A3}$$

A generally valid result from free-field theory reads, see [94]

$$\left\langle\exp\mathrm{i}\int\mathrm{d}u\,\mathrm{d}\boldsymbol{R}\,J(u,\boldsymbol{R})\phi(u,\boldsymbol{R})\right\rangle = \exp\left[-\frac{1}{2}\int\mathrm{d}u\,\mathrm{d}u'\,\mathrm{d}\boldsymbol{R}\,\mathrm{d}\boldsymbol{R}'\;J(u,\boldsymbol{R})\left\langle\phi(u,\boldsymbol{R})\phi(u',\boldsymbol{R}')\right\rangle J(u',\boldsymbol{R}')\right] \tag{A4}$$

which is one of the various forms of writing Wick's theorem. Using the explicit form of the current in (A3), equation (5.33) follows immediately.

A.2 The response function $\rho(t,s;\boldsymbol{r},\boldsymbol{r}')$

We now focus on the magnetic response function $\rho(t,s;\boldsymbol{r},\boldsymbol{r}')$. We decompose the MSR action

$$S[\phi,\widetilde{\phi},\mathbf{0}] = S_0[\phi,\widetilde{\phi}] + s[\phi,\widetilde{\phi}] \tag{A5}$$

where

$$\begin{aligned}S_0[\phi,\widetilde{\phi}] &= \int\mathrm{d}u\,\mathrm{d}\boldsymbol{r}\;\widetilde{\phi}\left[\frac{\partial\phi}{\partial u}-\Delta\phi\right]\\ s[\phi,\widetilde{\phi}] &= -\int\mathrm{d}u\,\mathrm{d}u'\,\mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{r}'\;\widetilde{\phi}(u,\boldsymbol{r})\kappa(u,u';\boldsymbol{r}-\boldsymbol{r}')\widetilde{\phi}(u',\boldsymbol{r}')\end{aligned} \tag{A6}$$

and

$$\kappa(u,u';\boldsymbol{r}-\boldsymbol{r}) = T\delta(u-u')\delta(\boldsymbol{r}-\boldsymbol{r}') + \frac{1}{2}\delta(u)\delta(u')a(\boldsymbol{r}-\boldsymbol{r}') \tag{A7}$$

The noiseless two-point functions are

$$\begin{aligned}
R_0(t,s;\boldsymbol{r},\boldsymbol{r}') &= \langle \phi(t,\boldsymbol{r})\widetilde{\phi}(s,\boldsymbol{r}')\rangle_0 = \Theta(t-s)[4\pi(t-s)]^{-d/2}\exp\left[-\frac{(\boldsymbol{r}-\boldsymbol{r}')^2}{4(t-s)}\right] \\
C_0(t,s;\boldsymbol{r},\boldsymbol{r}') &= \langle \phi(t,\boldsymbol{r})\phi(s,\boldsymbol{r}')\rangle_0 = 0 \\
\widetilde{C}_0(t,s;\boldsymbol{r},\boldsymbol{r}') &= \langle \widetilde{\phi}(t,\boldsymbol{r})\widetilde{\phi}(s,\boldsymbol{r}')\rangle_0 = 0
\end{aligned} \tag{A8}$$

Below, we shall need the equal-time response function $R(t,s;\boldsymbol{r},\boldsymbol{r}')|_{t=s}$. To give to this quantity a value, one may discretize the Langevin equation. This may be done according to several different schemes, see [59, 92]. Here, we shall use the Itô prescription which amounts to

$$R_0(t,t;\boldsymbol{r},\boldsymbol{r}') = 0 \tag{A9}$$

The magnetic response function reads from (A1)

$$\rho(t,s;\boldsymbol{r},\boldsymbol{r}') = \left\langle \widetilde{\phi}(s,\boldsymbol{r}')\sin(\phi(t,\boldsymbol{r})-\phi(s,\boldsymbol{r}'))\exp -s[\phi,\widetilde{\phi}]\right\rangle_0 \tag{A10}$$

where $\langle\rangle_0$ is the average with the $\exp\left(-S_0[\phi,\widetilde{\phi}]\right)$. Expanding the sine and using the Newton's binomial identity, we have

$$\begin{aligned}
\rho(t,s;\boldsymbol{r},\boldsymbol{r}') &= \sum_{k=0}^{\infty}\sum_{p=0}^{2k+1}\rho_{k,p}(t,s;\boldsymbol{r},\boldsymbol{r}') \\
\rho_{k,p}(t,s;\boldsymbol{r},\boldsymbol{r}') &= \frac{(-1)^{k+p+1}}{p!(2k+1-p)!}\left\langle \widetilde{\phi}(s,\boldsymbol{r}')\phi^p(t,\boldsymbol{r})\phi^{2k+1-p}(s,\boldsymbol{r}')\exp -s[\phi,\widetilde{\phi}]\right\rangle_0
\end{aligned} \tag{A11}$$

so that $\rho_{k,p}(t,s;\boldsymbol{r},\boldsymbol{r}')$ is given by a $2k+2$-point function, containing only one $\widetilde{\phi}$ contribution and $2k+1$ fields ϕ. The Bargman superselection rule (A8) implies that the only contractions that will lead to non-vanishing averages come from the k-th order in the expansion of the exponential, so that we have

$$\rho_{k,p}(t,s;\boldsymbol{r},\boldsymbol{r}') = \frac{(-1)^{p+1}}{p!(2k+1-p)!k!}\int\prod_{j=1}^{k}\mathrm{d}j\mathrm{d}j'\ \left\langle \widetilde{\phi}(s,\boldsymbol{r}')\phi^p(\boldsymbol{x},t)\phi^{2k+1-p}(s,\boldsymbol{r}')\widetilde{\phi}(j)\kappa(j,j')\widetilde{\phi}(j')\right\rangle_0 \tag{A12}$$

where for clarity the notation j, (respectively j') stands for $(u_j,\boldsymbol{r}_j)$, (respectively $(u_j^{'},\boldsymbol{r}_j^{'})$) and where integrals run over this set of $2k$ variables.

Wick's theorem states that the integrand decomposes into sums of products of two-point functions. In order for a contraction not to vanish, the field $\widetilde{\phi}(s,\boldsymbol{r}')$ must contract with one of the p fields $\phi(t,\boldsymbol{r})$, which leads to

$$\begin{aligned}
\rho_{k,0}(t,s;\boldsymbol{r},\boldsymbol{r}') &= 0 \\
\rho_{k,p}(t,s;\boldsymbol{r},\boldsymbol{r}') &= \frac{(-1)^{p+1}}{(p-1)!(2k+1-p)!k!}\int\prod_{j=1}^{k}\mathrm{d}j\mathrm{d}j'\ R_0(t,s;\boldsymbol{r},\boldsymbol{r}') \\
&\quad\times\left\langle \phi^{p-1}(t,\boldsymbol{r})\phi^{2k+1-p}(s,\boldsymbol{r}')\widetilde{\phi}(j)\kappa(j,j')\widetilde{\phi}(j')\right\rangle_0
\end{aligned} \tag{A13}$$

Summing over p, the response function $\rho(t,s;\boldsymbol{r},\boldsymbol{r}')$ reads

$$\rho(t,s;\boldsymbol{r},\boldsymbol{r}') = R_0(t,s;\boldsymbol{r},\boldsymbol{r}')\sum_{k=0}^{\infty}\frac{1}{(2k)!k!}\int\prod_{j=1}^{k}\mathrm{d}j\mathrm{d}j'\ \left\langle (\phi(t,\boldsymbol{r})-\phi(s,\boldsymbol{r}'))^{2k}\widetilde{\phi}(j)\kappa(j,j')\widetilde{\phi}(j')\right\rangle_0 \tag{A14}$$

At this stage, it will be interesting to give another equivalent expression of the magnetic correlation function $\Gamma(t,s;\boldsymbol{r},\boldsymbol{r}')$. Using the same strategy as before, but now expanding the cosine we find

$$
\begin{aligned}
\Gamma(t,s;\boldsymbol{r},\boldsymbol{r}') &= \sum_{k=0}^{\infty} \gamma_k(t,s;\boldsymbol{r},\boldsymbol{r}') \\
\gamma_k(t,s;\boldsymbol{r},\boldsymbol{r}') &= \frac{(-1)^k}{(2k)!} \left\langle (\phi(t,\boldsymbol{r})-\phi(s,\boldsymbol{r}'))^{2k} \exp -s[\phi,\widetilde{\phi}] \right\rangle_0
\end{aligned}
\tag{A15}
$$

For the same reason as before, because of mass conservation, only the expansion of order k of the exponential term will contribute and lead to non-vanishing contractions, such that

$$
\gamma_k(t,s;\boldsymbol{r},\boldsymbol{r}') = \frac{1}{(2k)!k!} \int \prod_{j=1}^{k} \mathrm{d}j\mathrm{d}j' \, \left\langle (\phi(t,\boldsymbol{r})-\phi(s,\boldsymbol{r}'))^{2k} \widetilde{\phi}(j)\kappa(j,j')\widetilde{\phi}(j') \right\rangle_0 \tag{A16}
$$

Comparing (A14) with (A16), we thus have

$$
\rho(t,s;\boldsymbol{r},\boldsymbol{r}') = R_0(t,s;\boldsymbol{r},\boldsymbol{r}')\Gamma(t,s;\boldsymbol{r},\boldsymbol{r}') \tag{A17}
$$

This is the main result of this appendix and gives eq. (5.38) in the text.

Appendix B. Scaling form of a special four-point function

We study the scaling of the four-point function

$$
\mathcal{F} := R_0^{(4)}(t,s,0,0;\mathbf{0},\mathbf{0},\boldsymbol{R},\boldsymbol{R}') = \left\langle \phi(t,\mathbf{0})\phi(s,\mathbf{0})\widetilde{\phi}(0,\boldsymbol{R})\widetilde{\phi}(0,\boldsymbol{R}') \right\rangle_0 \tag{B1}
$$

of a theory in MSR formulation of which the noiseless part is Schrödinger-invariant. The field ϕ is assumed to be quasiprimary with scaling dimension x and mass $\mathcal{M}$ and the response field $\widetilde{\phi}$ should also be quasiprimary with scaling dimension $\widetilde{x}$ and mass $\widetilde{\mathcal{M}} = -\mathcal{M}$. The covariance conditions on $\mathcal{R}$ are the following (we use $v(t)=0$)

$$
\begin{aligned}
\left(t\partial_t + s\partial_s + \frac{1}{2}\boldsymbol{R}\partial_{\boldsymbol{R}} + \frac{1}{2}\boldsymbol{R}'\partial_{\boldsymbol{R}'} + (x+\widetilde{x}) \right) \mathcal{F} &= 0 \\
\left(t^2\partial_t + s^2\partial_s + (t+s)x - \frac{\mathcal{M}}{2}\left(\boldsymbol{R}^2 + \boldsymbol{R}'^2\right) \right) \mathcal{F} &= 0
\end{aligned}
\tag{B2}
$$

We shall not discuss here the general solution of these equations. For our purposes it is enough to observe that if we decompose $\mathcal{F}$ in the following symmetrized way

$$
\mathcal{F} = \mathcal{G}(t,\boldsymbol{R})\mathcal{G}(s,\boldsymbol{R}') + \mathcal{G}(t,\boldsymbol{R}')\mathcal{G}(s,\boldsymbol{R}) \tag{B3}
$$

which for a free-field theory would follow from Wick's theorem, then

$$
\mathcal{G}(t,\boldsymbol{R}) = \mathcal{G}_0 t^{-x-\widetilde{x}} \exp\left[-\frac{\mathcal{M}}{2}\frac{\boldsymbol{R}^2}{t} \right] \tag{B4}
$$

produces a solution to both covariance conditions.

We apply this result to the preparation part C_{pr} of the autocorrelation function, with the intention to use the result in the spin-wave approximation of the XY model. From eq. (4.9) we have

$$
\begin{aligned}
C_{pr} &= \frac{1}{2}\int_{\mathbb{R}^{2d}} \mathrm{d}\boldsymbol{R}\mathrm{d}\boldsymbol{R}'\, a(\boldsymbol{R}-\boldsymbol{R}')R_0^{(4)}(t,s,0,0;\mathbf{0},\mathbf{0},\boldsymbol{R},\boldsymbol{R}') && \text{(B5)}\\
&= \frac{T_i}{(2\pi)^d}\int_{\mathbb{R}^{3d}} \mathrm{d}\boldsymbol{R}\mathrm{d}\boldsymbol{R}'\mathrm{d}\boldsymbol{q}\, \frac{(ts)^{-(x+\widetilde{x})/2}}{\boldsymbol{q}^2}\exp\left[\mathrm{i}\boldsymbol{q}\cdot(\boldsymbol{R}-\boldsymbol{R}')-\frac{\mathcal{M}}{2}\left(\frac{\boldsymbol{R}^2}{t}+\frac{\boldsymbol{R}'^2}{s}\right)\right] && \text{(B6)}\\
&= \int_0^\infty \mathrm{d}z\int_{\mathbb{R}^d}\frac{\mathrm{d}\boldsymbol{q}}{(2\pi)^d}T_i e^{-\boldsymbol{q}^2(t+s+z)} && \text{(B7)}\\
&= \frac{(4\pi)^{-d/2}}{d/2-1}T_i(t+s)^{1-d/2} && \text{(B8)}
\end{aligned}
$$

where in going to (B6) we used the initial condition (5.27) and the result (B3,B4) from above, next in going to (B7) we specialized to a free Gaussian field (where $x=\widetilde{x}=d$) of mass $\mathcal{M}=1/2$ and the last step we also assumed $d>2$.

Eq. (B8) provides the preparation term, for any initial temperature T_i, as required for the analysis of the autocorrelation function in the XY model in section 5.2

Appendix C. On local scale-invariance in the $1D$ Glauber-Ising model

C.1 Two-point functions in the $1D$ Glauber-Ising model

In the text we have seen that local scale-invariance implies the following form of the two-time autoresponse function

$$R(t,s) = r_0\,(t-s)^{-1-a}\left(\frac{t}{s}\right)^{1+a-\lambda_R/z} \tag{C1}$$

In spite of a nice agreement with a large variety of models, this expression is not verified for the $1D$ Ising model with Glauber dynamics at $T=0$.

The $1D$ Ising model is described by spins $\sigma_i=\pm1$ and the Hamiltonian

$$\mathcal{H} = -\sum_{i=1}\sigma_i\sigma_{i+1} \tag{C2}$$

The exactly solvable Glauber dynamics [36] may be given through the heat-bath rule, which gives the probability of finding the spin variables $\sigma_i(t+1)$ in terms of those at time t

$$P\left(\sigma_i(t+1)=\pm1\right) = \frac{1}{2}\left[1\pm\tanh\left(\beta\left(\sigma_{i-1}(t)+\sigma_{i+1}(t)+h_i(t)\right)\right)\right] \tag{C3}$$

where $\beta=1/T$ is the inverse temperature and $h_i(t)$ the external magnetic field. Ageing occurs in this model at $T=0$. In the long-time scaling limit, the two-time autocorrelation and autoresponse functions are [37, 65, 68]

$$
\begin{aligned}
R(t,s) &= \left.\frac{\delta\langle\sigma_i(t)\rangle}{\delta h_i(s)}\right|_{h=0} = \frac{1}{\pi\sqrt{2s(t-s)}} && \text{(C4)}\\
C(t,s) &= \langle\sigma_i(t)\sigma_i(s)\rangle = \frac{2}{\pi}\arctan\sqrt{\frac{2}{t/s-1}} && \text{(C5)}
\end{aligned}
$$

While these results were obtained first for a fully disordered initial state, they remain true for long-ranged initial conditions $\langle \sigma_r(0)\sigma_0(0)\rangle \sim r^{-\nu}$ with $\nu > 0$ [51]. The case $\nu = 0$ corresponds to the case of an initial magnetization m_0. Then the connected part of $C(t,s)$ as well as $R(t,s)$ are multiplied by $1 - m_0^2$ [77, 51]. In any case, the forms of the scaling functions $f_{C,R}(y)$ are unchanged.

Although these two-point functions clearly display dynamical scaling, it is evident that the scaling form of $R(t,s)$ from (C4) is incompatible with the form suggested in (C1). Local scale-invariance as developed in the text does not hold in the $1D$ Glauber-Ising model.

C.2 Generalized realization of the ageing algebra

We now show how Schrödinger invariance can be generalized such that the exact response function (C4) can be reproduced. Obviously, time-translation invariance is broken in ageing systems. Therefore, as already pointed out in [45, 46], the dynamical symmetry cannot be the Schrödinger Lie algebra $\mathfrak{sch}_1$ which contains the time-translation generator $X_{-1} = -\partial_t$, but a subalgebra without this generator might be acceptable. We consider the algebra [48]

$$\mathfrak{age}_1 := \{X_0, X_1, Y_{-1/2}, Y_{1/2}, M_0\} \tag{C6}$$

and keeping the commutation relations (2.7) we now look for a more general realization of $\mathfrak{age}_1$. In this way, we write the generators as $\{\Xi_{0,1}, \Upsilon_{\pm 1/2}, M_0\}$. These must be of the form

$$\begin{aligned} \Xi_0 &= -t\partial_t - \frac{1}{2}r\partial_r - \frac{x}{2} \\ \Xi_1 &= -t^2\partial_t - tr\partial_r - xt - g(t) - \frac{\mathcal{M}}{2}r^2 \\ \Upsilon_{-1/2} &= -\partial_r \\ \Upsilon_{1/2} &= -t\partial_r - \mathcal{M}r \\ M_0 &= -\mathcal{M} \end{aligned} \tag{C7}$$

where $g = g(t)$ is to be found. The only commutator of $\mathfrak{age}_1$ constraining g is

$$[\Xi_1, \Xi_0] = \Xi_1 \tag{C8}$$

which leads to

$$t\partial_t g - g = 0 \tag{C9}$$

with the solution $g(t) = Kt$, with K some constant. From (C7), the dynamical exponent $z = 2$. If we were to require in addition $[\Xi_1, \Xi_{-1}] = 2\Xi_0$ (and thereby go from $\mathfrak{age}_1$ back to $\mathfrak{sch}_1$) we would recover $K = 0$. Now, a quasiprimary field ϕ of $\mathfrak{age}_1$ will be characterized by a triplett $(x, K, \mathcal{M})$.

We can now generalize local scale-invariance by requiring that the autoresponse function $R(t,s)$ formed from a quasiprimary field ϕ and its associated quasiprimary response field $\widetilde{\phi}$ to transform covariantly under the generators Ξ_0 and Ξ_1. It is a solution of the system of linear partial differential equations

$$\begin{aligned} \left[t\partial_t + \frac{x}{2} + s\partial_s + \frac{\widetilde{x}}{2}\right] R(t,s) &= 0 \\ \left[t^2\partial_t + (K+x)t + s^2\partial_s + (\widetilde{K} + \widetilde{x})s\right] R(t,s) &= 0 \end{aligned} \tag{C10}$$

where $\widetilde{x}$ and $\widetilde{K}$ refer to the response field $\widetilde{\phi}$. Solving the system (C10) gives as final result

$$
\begin{aligned}
R(t,s) &= s^{-1-a} f_R(t/s) \\
f_R(y) &= r_0 y^{1+A-\lambda_R/2} (y-1)^{-1-A}
\end{aligned}
\tag{C11}
$$

where the three independent non-equilibrium exponents a, A, λ_R are

$$
\begin{aligned}
a &= \frac{x+\widetilde{x}}{2} - 1 \\
A &= a + K + \widetilde{K} \\
\lambda_R &= 2x + 2K
\end{aligned}
\tag{C12}
$$

In contrast with the previous realization of $\mathfrak{age}_1$, $K, \widetilde{K} \neq 0$ is possible and then a and A differ from each other.

Comparison with the exact result (C4) of the $1D$ Glauber-Ising model now gives complete agreement and we identify the exponents

$$
a = 0 \ \ , \ \ A = -\frac{1}{2} \ \ , \ \ \lambda_R = 1
\tag{C13}
$$

Of course, the values of a and λ_R have been obtained before [37, 65] but A seems to be a new exponent.

At present, it must remain open whether the unusual properties of the $1D$ Glauber-Ising model are related to the fact that $T_c = 0$ and therefore the critical and low-temperature properties might have become mixed.

Also, it remains to be seen whether the form of the autocorrelation function can be understood from the generalized realization of the ageing algebra $\mathfrak{age}_1$. We hope to come back to this elsewhere.

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